Long Range Order in Random Field Ising Model

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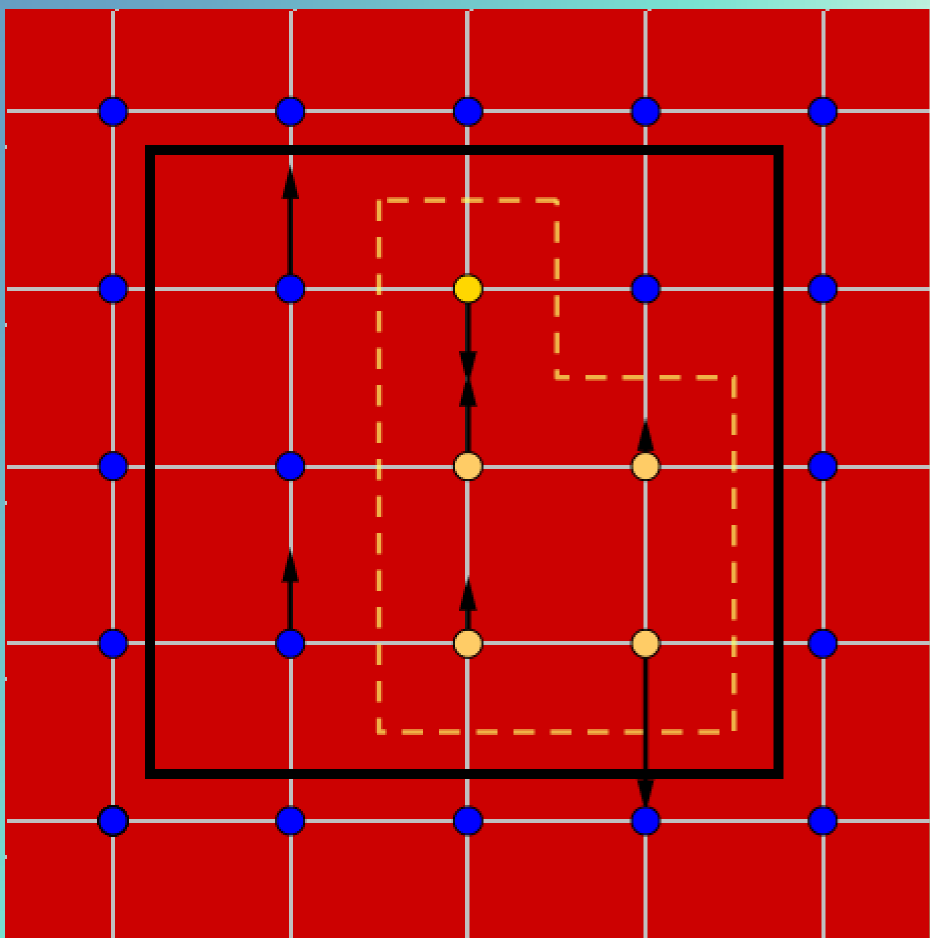
January 2022



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1 Classical Lattice Systems

Statistical mechanics deals with the task of trying to make macroscopic quantities (usually thermodynamic ones) emerge from the prescription of a very large system of microscopic entities and the interactions between them. It turns out that those macroscopic quantities do not depend sharply on the explicit details of the microscopic state, but arise as averages in the set of such "microstates". So, in order to achieve our goal, it is necessary to know the probability of occurrence of each microstate. Well known physical arguments may be used to loosely "derive" this distribution, named as Gibbs distribution, which tends to be proportional to $e^{-\beta H}$, where β is the inverse of the temperature and H is the Hamiltonian, that is, the energy of the microstate, as will be better developed later.

One of the most important features of real systems, however, that we would like to reproduce is the phenomenon of phase transition, which usually is manifested by the non-analyticity of a function called pressure. It turns out that systems of finite particles have analytical pressure and thus lack genuine phase transitions. The critical procedure is to take the so called thermodynamic limit, that is, to take the limit of the number of particles going to infinity, in order to the critical behavior become apparent.

Among the models of microscopic systems, the easiest ones to mathematically deal with are the lattice systems[1]. This case consists of a set of particles, each one lying at a site of a lattice $\mathcal{L}$, usually taken to be $\mathcal{L} = \mathbb{Z}^d$ (endowed with the L^1 norm), in contrast to continuum systems, where the scenario is usually taken to be $\mathbb{R}^d$. In the classical setting (the one that we will treat in this text), each particle/site has a set of possible states called state space. Those systems can model more realistically, for example, crystal structures of atoms with spins. By its simplicity, it is one of the most developed branches and the Ising model, being the most studied one in all statistical mechanics, will be our focus. This model has $\{-1, 1\}$ as state space corresponding to "spin up" and "spin down" and was proposed around 1920 by the physicist Lenz for the PhD thesis of one of his students, Ising, as a simplified way of dealing with ferromagnetic systems.

1.1 State Space

Before diving deeper in the Ising model, we are going to talk a little about classical lattice systems in more generality. The state space will be a probability[2] space $(\mathcal{S}_0, \mathcal{F}_0, \nu)$ where $\mathcal{S}_0$ is a polish space, $\mathcal{F}_0$ is the borel sigma-algebra and ν is called "a priori distribution". In most cases, $\mathcal{S}_0$ is a finite set with the discrete metric, $\mathcal{F}_0 = 2^{\mathcal{S}_0}$ and ν is the uniform measure. This will always be the case in the following examples where $\mathcal{S}_0$ is finite. The examples of state spaces presented below refer to models that are actually of interested in the area. We remark that to complete describe a model we should also offer the hamiltonian, which will be done later. We assume that $\mathcal{L} = \mathbb{Z}^d$ in every example.

[1] We mention *en passant* that lattice systems may be generalized a little further replacing the lattice with some arbitrary graph.

[2] For the sake of completeness and to please the more demanding readers, we say that there are relevant situations where the measure space in question is not a probability one. For example, we may have $(\mathbb{R}, \mathcal{B}, \lambda)$

Examples - State Space

Example 1.1. *(Ising Model) As already discussed, $\mathcal{S}_0 = \{-1, 1\}$, corresponding to spin up and spin down states.*

Example 1.2. *(Lattice Gas) $\mathcal{S}_0 = \{0, 1\}$. In this model, each vertex of the lattice must be seen as a site where may or may not have some particle. Naturally, the state 0 stands for no occupation and 1 stands for a site occupied. As one might imagine, the state space is "isomorphic" to the Ising space, and there is in fact huge similarities in these two models*

Example 1.3. *(Potts Model) Slightly generalization of the Ising model taking account of a system with q spins. Viewing strictly as a spin model, the state space should be rigorously something like ${\mathcal{S}_0}^q = \{(\cos(\theta_n), \sin(\theta_n)) \in \mathbb{R}^2; \theta_n = 2\pi n/q\}$. But, more generally (and to simplify the things a bit) we will regard the space just abstractly as ${\mathcal{S}_0}^q = \{1, ..., q\}$.*

Example 1.4. *(n-vector model) In this model $\mathcal{S}_0$ is the $(n-1)-$sphere of $\mathbb{R}^n$ where the a priori measure is Lebesgue (naturally, with the borel $\sigma-$algebra). In the case where $n = 1$ this model is just Ising, in $n = 2$ it is called XY model and for $n = 3$ we call it Heisenberg model.*

Example 1.5. *(Gaussian Free Field Model) In this case $\mathcal{S}_0 = \mathbb{R}$, $\mathcal{F}_0$ is the usual borel $\sigma-$algebra and the a priori measure has gaussian distribution. This model can be generalized in order to get $\mathcal{S}_0 = \mathbb{R}^N$*

Example 1.6. *The discrete gaussian model and the solid-on-solid model (SOS) uses $\mathcal{S}_0 = \mathbb{Z}$, with $\mathcal{F}_0 = 2^{\mathbb{Z}}$.*

1.2 Configuration Space

The configuration space $(\mathcal{S}, \mathcal{F})$ is taken to be $\mathcal{S} = (\mathcal{S}_0)^{\mathcal{L}}$ endowed with the product topology and $\mathcal{F}$ being the product $\sigma-$algebra, so we are going to spend a little time reviewing the concept of a product $\sigma-$algebra and product spaces in general. This theme is rather abstract and quite confusing, but I will do my best to explain and try to picture the concepts and I hope you will do your best to understand what I am saying.

We will start by considering the most general case, so let $(\Omega_i, \mathcal{F}_i)_{i \in I}$ be a family of measurable spaces.

1.2.1 Cylinder Sets

A rectangular cylinder in Ω with base $J \subset I$ is a subset C of the form $C = \bigtimes_{i \in I} A_i$, where $A_i \in \mathcal{F}_i$ and $A_i = \Omega_i$ for every $i \notin J$. There is an even more general notion of cylinder that does not need to be rectangular, but every cylinder we will consider in this text will be a rectangular one, so we will refer to them simply as "cylinder" without risk of misunderstanding. Notice that:

$$\left(\bigtimes_{i\in I} A_i\right) \cap \left(\bigtimes_{i\in I} B_i\right) = \bigtimes_{i\in I} A_i \cap B_i$$

Thus, the set of all cylinders with finite base is $\cap-$closed. But rarely a set of cylinder is $\cup-$closed. Consider, for example, $\Omega = E^{\mathbb{N}}$ with E as simple as $\{-1, 1\}$. We cannot write the union of $\{-1\} \times E \times E \times ...$ with $E \times \{1\} \times E \times E \times ...$ as one unique (rectangular) cylinder. About the complement, we have:

$$\left(\bigtimes_{i\in I} A_i\right)^c = \bigcup_{j\in J} \left(\bigtimes_{i\in I} A_i^j\right)$$

Where $A_i^j = (A_i)^c$ if $i = j$ and Ω_i otherwise. Adding the two statements together, we obtain that:

$$\left(\bigtimes_{i\in I} A_i\right) \setminus \left(\bigtimes_{i\in I} B_i\right) = \bigcup_{j\in J} \left(\bigtimes_{i\in I} A_i \setminus (B_i^j)^c\right)$$

Where J is the base of $\bigtimes_{i\in I} B_i$.

Actually, we will be more interested in the cylinders with a finite base. In this case, the facts stated above implies that the set of the finite unions of cylinders with finite base forms a algebra of sets.

1.2.2 Product $\sigma-$algebras

The product $\sigma-$algebra is defined as the smallest $\sigma-$algebra in $\Omega := \bigtimes_{i\in I} \Omega_i$ that makes all the projections measurable.

The product $\sigma-$algebra is generated by the set of all rectangular cylinders with finite base. Thus, in the case of a family of countable sets, it suffices to consider the cylinders $\bigtimes_{i\in I} A_i$, where $A_i = \Omega_i$ for every i with exception of one, where A_i is unitary. The following theorem also holds [6]:

Theorem 1.1: [6]

If I is countable, each Ω_i is a polish space and $\mathcal{F}_i$ is its correspondent Borel $\sigma-$algebra, then the product space is polish and the product $\sigma-$algebra coincides with the one generated by the product topology. In other words:

$$\bigtimes_{i\in I} \sigma(\tau_i) = \sigma\left(\bigtimes_{i\in I} \tau_i\right)$$

Thus, $\mathcal{F}$ may be simply regarded as the borel $\sigma-$algebra in $\mathcal{S}$.

We also define $\Omega_\Lambda := \bigtimes_{\ell\in\Lambda} \Omega_\ell$, for any set $\Lambda \subset I$ and $\mathcal{F}_\Lambda$, $\Lambda \subset I$ as the $\sigma-$algebra generated by the (rectangular) cylinders with (finite) base $\Lambda \subset I$. If $E \in \mathcal{F}_\Lambda$, then E only depends on the coordinates of Λ, i.e., the restrictions must be made only on the coordinates of Λ, being the rest "free".

One way to rigorously make sense of "only depends on the coordinates of Λ" is the following. Let 2^Ω_Λ be the collection $\{E \in 2^\Omega;\ \pi_i(E) = \Omega_i,\ \forall i \notin \Lambda\}$, π_i being the $i-$th projection. If, for example, $I = \mathbb{N}$ and $\Lambda = \{i, i+1, ..., i+n\}$, then the sets of 2^Ω_Λ are of the form[3]:

$$E = \Omega_1 \times ... \times \mathscr{A} \times \Omega_{i+n+1} \times ...$$

Where $\mathscr{A}$ is *any* subset of $\Omega_i \times ... \times \Omega_{i+n}$. This is the collection of sets that only depend on the coordinates of Λ. Now, just notice that 2^Ω_Λ is a $\sigma-$algebra and it contains all the rectangular cylinders with base $J \subset \Lambda$, so $\mathcal{F}_\Lambda \subset 2^\Omega_\Lambda$

With this picture in mind, we may remark something relevant: $\mathcal{F}_\Lambda$ may be identified with $\bigotimes_{\ell\in\Lambda} \mathcal{F}_\ell$, although the elements of the latter are in $\bigtimes_{\ell\in\Lambda} \Omega_\ell$ and the elements of $\mathcal{F}_\Lambda$ are in $\bigtimes_{\ell\in\Lambda} \Omega_\ell$. Similarly, we can identify measures in $(\bigtimes_{\ell\in\Lambda} \Omega_\ell, \bigotimes_{\ell\in\Lambda} \mathcal{F}_\ell)$ with measures in $(\Omega, \mathcal{F}_\Lambda)$.

The way to formalize this identification is by means of the generalized projection maps $\pi_\Lambda : \Omega \to \bigtimes_{\ell\in\Lambda} \Omega_\ell$ be the projection: $\pi_\Lambda(\omega) = (\omega_\ell)_{\ell\in\Lambda}$. If μ is a measure in $(\Omega, \mathcal{F}_\Lambda)$, if μ is a measure in Ω, we may define a measure μ_Λ as the pushforward: $\mu_\Lambda(E) = \mu(\pi_\Lambda^{-1}(E))$, for $E \in \bigotimes_{\ell\in\Lambda} \mathcal{F}_\ell$.

A sequence $(\Lambda_n)_n$ is called absorbing if it is increasing and the union coincides with $\mathcal{L}$. It is denoted by $\Lambda_n \uparrow \mathcal{L}$

1.3 Observables

Observables are just measurable functions. An observable f is called local if there is a finite subset $\Lambda \subset \mathcal{L}$ such that $f(\omega) = f(\omega')$ as long as $\omega_\Lambda = \omega'_\Lambda$, that is, $\omega_i = \omega'_i$ for every $i \in \Lambda$. An observable is quasilocal if it is the uniform limit of local functions, or, what is equivalent, the following limit holds:

$$\lim_{\Lambda\uparrow\mathcal{L}} \sup_{\omega_\Lambda = \omega'_\Lambda} |f(\omega') - f(\omega)| = 0$$

Proof. ■

The space of bounded functions $B(\mathcal{S})$ is a Banach space with respect to the supremum norm and it is also true for the space of quasilocal bounded functions, $B_{ql}(\mathcal{S})$, the space of continuous bounded functions $C_b(\mathcal{S})$ and the space of quasilocal continuous bounded functions, $C_{b,ql}(\mathcal{S})$. This is not necessarily true if we consider only local but not quasilocal functions, such as $B_{loc}(\mathcal{S})$ or $C_{b,loc}(\mathcal{S})$.

[3]This type of set is more similar to the *general* cylinder. The only difference is that in the general cylinder we don't allow $\mathscr{A}$ to be any, but has to be "measurable" in some sense ($\mathscr{A} \in \bigotimes_{\ell\in\Lambda} \mathcal{F}_\ell$)

1.4 Interactions and Hamiltonians

Definition 1.1: Interactions and their classes

An interaction is a family $\Phi = \{\Phi_A\}_{A\subset\mathcal{L}}$ such that $\Phi_A \in B(\mathcal{S}, \mathcal{F}_A)$, that is, Φ_A is bounded and local with respect to A. An interaction is called regular, or absolutely summable if, for every $x \in \mathcal{L}$ there exists c such that:

$$\sum_{A\ni x} ||\Phi_A||_\infty \leq c$$

And it is called uniformly absolutely summable if:

$$|||\Phi||| = \sup_{x\in\mathcal{L}} \sum_{A\ni x} ||\Phi_A||_\infty < \infty$$

The set of such interactions forms a Banach space, while the set of regular interactions forms a Frechet space.

Given an interaction, one can associate an hamiltonian to it. The hamiltonian with free boundary condition is given by:

$$H^\Phi_{f,\Lambda}(\sigma) = \sum_{A\subset\Lambda} \Phi_A(\sigma)$$

Where "f" stands for "free", although the most useful (and the one we will use) will be the following:

$$H^\Phi_\Lambda(\sigma) = \sum_{A\cap\Lambda\neq\emptyset} \Phi_A(\sigma)$$

In the following we are going to give relevant examples of interactions in the theory. Notice that all examples will have at most 2$-$body interactions, that is, $\Phi_A \equiv 0$ if $\#A > 2$

Examples - Interactions

Example 1.7. *(Ising-type Models) The interaction is given by $\Phi_A(\sigma) = -h_i\sigma_i$ if $A = \{i\}$, $\Phi_A(\sigma) = -J_{ij}\sigma_i\sigma_j$ if $A = \{i,j\}$ and 0 otherwise, where $(J_{ij})_{i,j\in\mathbb{Z}^d}$ and $(h_i)_{i\in\mathbb{Z}^d}$ are families of constants, with the later being called "external field". Later we give a more detailed motivation of this interaction. The hamiltonian becomes:*

$$H_\Lambda(\sigma) = -\sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - \sum_{i\in\Lambda, j\notin\Lambda} J_{ij}\sigma_i\sigma_j - \sum_{i\in\Lambda} h_i\sigma_i$$

There are some subtypes that are worthwhile to mention. The model is said to have "nearest-neighbour" interaction if $J_{ij} = J$ for i,j such that $d(i,j) = 1$ and zero otherwise. In the case that $J_{ij} \neq 0$ for i,j arbitrarily far apart from each other we call the model "long range". Moreover, the model is called ferromagnetic if $J_{ij} > 0$, $\forall i,j$ and antiferromagnetic if $J_{ij} < 0$, $\forall i,j$.

Example 1.8. *(Potts Model) We have* $\Phi_A(\sigma) = -h_i(\sigma_i)$ *if* $A = \{i\}$*, where* (h_i) *is a family of functions on* $\{1, ..., q\}$*,* $\Phi_A(\sigma) = -J_{ij}\delta(\sigma_i, \sigma_j)$ *if* $A = \{i, j\}$*, where* δ *stands for the Kronecker delta and* $\Phi_A \equiv 0$ *otherwise. We will also justify better this model later and the same considerations made about the Ising model apply here: we may have a nearest-neighbour model, a finite range and a long range one, as well as ferromagnetic or antiferromagnetic.*

Example 1.9. *(Inhomogeneous Potts Model) Instead of letting the coupling constant* J *depends upon the pair of sites, we may let it depends upon the spins, giving rise to the inhomogeneous Potts model. For the sake of simplicity, we will consider the nearest-neighbour case. In this case,* $\Phi_A(\sigma) = \sum_{r,r' \in \{1,\ldots,q\}} J_{r,r'}\delta(\sigma_i, r)\delta(\sigma_j, r')$*, for* $A = \{i, j\}$ *and* $d(i, j) = 1$*,* Φ_A *equals to the previous example if* $\#A = 1$ *and* 0 *otherwise.*

Example 1.10. *(SOS and discrete gaussian model) The 2-body interaction is given by* $\Phi_A(\sigma) = |\sigma_i - \sigma_j|^\alpha$*, where* $A = \{i, j\}$ *and* $d(i, j)$*. Along the line of the previous examples, some changes may be made in order to define a long-range Hamiltonian, an external field, etc. The model is called SOS if* $\alpha = 1$ *and discrete gaussian if* $\alpha = 2$*.*

1.5 Measures

Consider the space of finite signed measures of Ω, $M(\Omega)$. There are several different topologies we may put on this space. We may endow this space with the uniform topology defined by:

$$||\mu - \nu|| = \sup_{||f||_\infty \leq 1} |\mu(f) - \nu(f)|$$

Where the functions in the supremum is usually taken in the set $B(\mathcal{S})$. This notion of convergence, however, is exaggeratedly strong, so the most useful ones is of the following kind: $\mu_n \to \mu$ if $\mu_n(f) \to \mu(f)$ for every f in a certain class. Notice that we do not demand the convergence to be uniform in f. The different topologies will arise from different classes of functions. The typical classes are $B(\mathcal{S}), B_{ql}(\mathcal{S}), C_b(\mathcal{S}), C_{b,ql}(\mathcal{S})$, that where already presented.

The following sections will be devoted to the study of special classes of measures that have connection to the physics of statistical mechanics problems.

1.5.1 Finite Gibbs Measures

As already mentioned, the physical relevant measures are those which are proportional to $e^{-\beta H}$, and are called Gibbs measure. This statement, however, only makes sense for finite spaces, once H_Λ is only well-defined in this case, and will be the theme of this section. We are going to do some extra work to find the suitable measures in the infinite case, in the next sections.

To better deal with finite subsystems we are going to introduce some useful notation. In first place, from now on, Λ will represent a finite subset of the lattice $\mathbb{Z}^d$. An element of $\mathcal{S}_A$

will be frequently denoted by σ_A. If $\sigma_A \in \mathcal{S}_A$ and $\omega_{A^c} \in \mathcal{S}_{A^c}$, we usually denote by $\sigma_A\omega_{A^c}$ the configuration of $\mathcal{S}$ which coincides with σ in A and with ω outside A.

Thus, the finite Gibbs measure at inverse temperature β and boundary condition $\omega \in \mathcal{S}$ is defined, for every $A \in \mathcal{F}$, by:

!

$$\mu^{\omega}_{\Lambda,\beta}(A) := \frac{1}{Z^{\omega}_{\Lambda,\beta}} \int_{\mathcal{S}_\Lambda} \chi_A(\sigma_\Lambda\omega_{\Lambda^c}) e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda\omega_{\Lambda^c})} d\nu_\Lambda(\sigma_\Lambda)$$

$$Z^{\omega}_{\Lambda,\beta} := \int e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda\omega_{\Lambda^c})} d\nu_\Lambda(\sigma_\Lambda)$$

Where ν_Λ is the product measure $\bigotimes_{i\in\Lambda} \nu_i \in \mathcal{S}_\Lambda$. But notice that, the way we have defined, $\mu^{\omega}_{\Lambda,\beta}$ are measures in the whole configuration space $\mathcal{S}$, not only in $\mathcal{S}_\Lambda$.

Quantities like $Z^{\omega}_{\Lambda,\beta}$ above are called "partition functions" and introduced in order to normalize the measure. However, although being introduced merely as a normalization factor, they play a completely unreasonable and unexpected fundamental role in the theory.

For free boundary condition we have, for $A \subset \Omega_\Lambda$

$$\mu^{\varnothing}_{\Lambda,\beta}(A) := \frac{1}{Z^{\varnothing}_{\Lambda,\beta}} \int_{\mathcal{S}_\Lambda} \chi_A(\sigma_\Lambda) e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)} d\nu_\Lambda(\sigma_\Lambda) = \frac{1}{Z^{\varnothing}_{\Lambda,\beta}} \int_{A} e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)} d\nu_\Lambda(\sigma_\Lambda)$$

$$Z^{\varnothing}_{\Lambda,\beta} := \int e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)} d\nu_\Lambda(\sigma_\Lambda)$$

Or, in a more simplified way:

$$d\mu^{\varnothing}_{\Lambda,\beta}(\sigma_\Lambda) := \frac{1}{Z^{\varnothing}_{\Lambda,\beta}} e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)} d\nu_\Lambda(\sigma_\Lambda)$$

In the very relevant case where $\mathcal{S}_0$ is finite and the *a priori* measure is uniform, one can gets rid of the integrals with respect to ν, as we argue now.

In first place, we remark that, in this case, $\mathcal{S}_\Lambda$ is finite and ν_Λ is simply the uniform measure in $(\mathcal{S}_\Lambda, \mathcal{F}_\Lambda)$, as may be easily shown by a straightforward calculation. So, without loss of generality, we only need to know $\mu^{\varnothing}_{\Lambda,\beta}(\eta)$, for arbitrary $\eta \in \mathcal{S}_\Lambda$. We have:

$$\mu^{\varnothing}_{\Lambda,\beta}(\eta) = \frac{1}{Z^{\varnothing}_{\Lambda,\beta}} \int_{\mathcal{S}_\Lambda} \chi_{\{\eta\}}(\sigma_\Lambda) e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)} d\nu_\Lambda(\sigma_\Lambda) = \frac{e^{-\beta H^{\Phi}_{\Lambda}(\eta)} \nu_\Lambda(\eta)}{Z^{\varnothing}_{\Lambda,\beta}} = \frac{1}{\#\Omega_\Lambda} \frac{e^{-\beta H^{\Phi}_{\Lambda}(\eta)}}{Z^{\varnothing}_{\Lambda,\beta}}$$

But we then have:

$$Z^{\varnothing}_{\Lambda,\beta} = \sum_{\sigma_\Lambda \in \mathcal{S}_\Lambda} e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)} \nu_\Lambda(\sigma_\Lambda) = \frac{1}{\#\mathcal{S}_\Lambda} \sum_{\sigma_\Lambda \in \Omega_\Lambda} e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)}$$

Finally giving us:

$$\mu^{\varnothing}_{\Lambda,\beta}(\eta) = \frac{e^{-\beta H^{\Phi}_{\Lambda}(\eta)}}{\sum_{\sigma_\Lambda \in \mathcal{S}_\Lambda} e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)}}$$

In the case of finite state space (and uniform a priori measures) we usually define the partition function without the factor $\#\mathcal{S}_\Lambda$, so $Z_{\Lambda,\beta} := \sum_{\sigma_\Lambda \in \mathcal{S}_\Lambda} e^{-\beta H^{\Phi}_{\Lambda}(\sigma_\Lambda)}$ and we have:

!

$$\mu^{\varnothing}_{\Lambda,\beta}(\sigma) = \frac{e^{-\beta H^{\Phi}_{\Lambda}(\sigma)}}{Z^{\varnothing}_{\Lambda,\beta}}$$

Which is a very well-known formula.

The reasoning is the same for measures with boundary condition.

1.5.2 DLR measures and Specifications

We are going to make the first attempt in defining infinite volume Gibbs measure. The idea, due to Dobrushin, Ruelle and Lanford (that's the reason for the name DLR), is gluing each finite measure with one another. Stating another way, a DLR measure will be such that it has to be equal to the finite volume Gibbs measure if we restrict our attention to a particular finite region of the lattice. The way to translate this concept into a rigorous statement is through the usage of *conditional expectations*.

Definition 1.2: DLR measure

Let Φ be a regular interaction. We say that a measure in $(\mathcal{S}, \mathcal{F})$ is a DLR measure for Φ if, for $\mu-$qtp ω:

$$\mathbb{E}_\mu(A|\mathcal{F}_{\Lambda^c})(\omega) := \mathbb{E}_\mu(\chi_A|\mathcal{F}_{\Lambda^c})(\omega) = \mu^{\omega}_{\Lambda,\beta}(A),\ \forall A \in \mathcal{F}, \forall \Lambda \in \mathcal{P}_f(\mathbb{Z}^d)$$

Where $\mu_{\Lambda,\beta}$ is obviously the finite volume Gibbs measure for the interaction Φ.

In this section, we will be interested in seeing the finite volume Gibbs measures as functions both of the set we are measuring and of the boundary condition, so we are going to adapt the notation a little. We will write $\mu_{\Lambda,\beta}(\omega, A)$ instead of $\mu^{\omega}_{\Lambda,\beta}(A)$. $\mu_{\Lambda,\beta}$ will be, then, a family of probability measures indexed by $\omega \in \mathcal{S}$. Moreover, we have the following result:

Proposition 1.1

The map $\omega \mapsto \mu_{\Lambda,\beta}(\omega, A)$ is measurable for every $A \in \mathcal{F}_\Lambda$

The previous proposition implies that $\mu_{\Lambda,\beta}$ are probability kernels:

Definition 1.3: Probability Kernels

Let $(\Omega_1, \mathcal{F}_1)$ and $(\Omega_2, \mathcal{F}_2)$ be measurable spaces. A function $\pi : \Omega_1 \times \mathcal{F}_2 \to [0, 1]$ is called a probability kernel if the following properties holds:

1. $\pi(\cdot, A)$ is a $(\Omega_1, \mathcal{F}_1)-$measurable function, $\forall \in \mathcal{F}_2$.
2. $\pi(x, \cdot)$ is a probability measure in $(\Omega_2, \mathcal{F}_2)$

More precisely, $\mu_{\Lambda,\beta}$ are probability kernels from $(\Omega_1, \mathcal{F}_1) = (\mathcal{S}, \mathcal{F}_{\Lambda^c})$ to $(\Omega_2, \mathcal{F}_2) = (\mathcal{S}, \mathcal{F})$.

Probability kernels are ubiquitous in probability theory and related areas and it is a very important and useful concept. One may understand it exactly as a family of probability measures depending on the point of some space. Actually, there are more properties of finite volume Gibbs measures besides being a probability kernel that are important. For example, it is straightforward to see that $\mu_{\Lambda,\beta}(\cdot, A) = \chi_A$, for every $A \in \mathcal{F}_{\Lambda^c}$. Probability kernels that satisfy this property are called *proper*, and this condition is equivalent to saying that $\pi(\cdot, A \cap B) = \chi_A \pi(\cdot, B)$ for every $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$.

Furthermore, we have:

Proposition 1.2: Consistence Condition

Let $\Lambda \subset \Delta$. Then:

$$\int_{\mathcal{S}} \mu_{\Lambda,\beta}(\eta, A) d\mu^{\omega}_{\Delta,\beta}(\eta) = \mu_{\Delta,\beta}(\omega, A)$$

Which is known as consistence condition. The integral may be abbreviated to $\mu^{\omega}_{\Delta}\mu^{(\cdot)}_{\Lambda} = \mu^{\omega}_{\Delta} \equiv \mu_{\Delta}\mu_{\Lambda} = \mu_{\Delta}$. The last proposition tells us that the Gibbs measures are specifications:

Definition 1.4: Specifications and Gibbsian Specifications

A specification $\gamma = \{\gamma_\Lambda\}_{\Lambda \in \mathcal{P}_f(\mathbb{Z}^d)}$ is a family of probability kernels from $(\mathcal{S}, \mathcal{F}_{\Lambda^c})$ to $(\mathcal{S}, \mathcal{F})$ satisfying:

1. γ_Λ is $\mathcal{F}_{\Lambda^c}$ proper, that is, $\gamma_{\Lambda,\beta}(\cdot, A) = \chi_A$, for every $A \in \mathcal{F}_{\Lambda^c}$
2. For all $\Lambda, \Delta \in \mathcal{P}_f(\mathbb{Z}^d)$ with $\Lambda \subset \Delta$, it holds the consistence condition:

$$\gamma^{\omega}_{\Delta}\gamma^{(\cdot)}_{\Lambda} = \gamma^{\omega}_{\Delta}$$

A specification γ is called gibbsian if there is an interaction Φ and $\beta \leq 0$ such that $\gamma_\Lambda = \mu_{\Lambda,\beta}$ for every Λ, where $\mu_{\Lambda,\beta}$ is obviously the finite volume Gibbs measure for the interaction Φ.

1.5.3 Equilibrium measures

The idea of equilibrium measures comes from the principle of physics that the state achieved by a system will be the one (or the ones) that minimizes the free energy function, which is given by $F = U - TS$, where U is the average energy, T the temperature and S the entropy. Multiplying everything by $-\beta$, the "equilibrium states" will be those that maximizes the quantity $S - \beta U$. Now we are left with the following question: what in Earth will play the role of U and S? This is a subtle question left to be treated.

2 Ising Model

In order to reproduce the ferromagnetic behavior, it is necessary to allow one particle spin to influences the spin of adjacent particles [4]. In the first models studied, each particle could interact only with its nearest neighbours. Later, more general models were considered (e. g. the so-called “long range Ising models"), but we are sticking with the first ones in this essay. One of the things that made this model so famous was the discover of phase transitions for dimension greater that one by an ingenious argument of Peierls in 1936.

As already presented, the hamiltonian in a finite subset Λ in its full glory is given by:

$$H(\eta, \Lambda_N, h; \sigma) = -J \sum_{u \sim v, u, v \in \Lambda_N} \sigma_u \sigma_v - J \sum_{u \sim v, u \in \Lambda_N, v \notin \Lambda_N} \sigma_u \eta_v - \sum_{u \in \Lambda_N} h_u \sigma_u$$

Where $u \sim v$ if u and v are nearest neighbours, that is, if $||u-v||_{L^1} = 1$ and we are assuming $J > 0$ (a justification for this restriction will be made above).

A few commentaries are due. In first place h is to be interpreted as a magnetic field external to the system and h_u its value on u. Furthermore, we point out that, in principle, the system is supposed to assume the state of least energy. Suppose for a moment that $h_u = 0$ for every u. Once $J > 0$, the bigger $\sigma_u \sigma_v$, the lesser the energy. As the spin can be only 1 or -1, this is accomplished once $\sigma_u = \sigma_v$, that is, one spin has the tendency to be equal to its neighbours, in order to minimize the energy. Actually, the (two) configurations where every particle has the same spin are the ones that achieve the minimum. These are called ground states. These configurations are so important that we will call them special names: σ^+ or simply by $+$ for σ such that $\sigma_i = +1$ for every $i \in \mathbb{Z}^d$ and σ^- and $-$ for the other one. Naively, one might think that these are the most likely states, and it is in fact the case when the temperature is zero. However, $T > 0$ means that we have some degree of disorder in our system, so the answer is not so obvious in this case. Notice that all this discussion is possible since $J > 0$, and that is the reason why we call this condition on J “ferromagnetic". If this was not the case, however, the ground states would be the ones with alternate spins.

Coming back to the field issue, there are some well-studied cases at hand. The first one, for example, is the case where the magnetic field is constant. There is also the case where we have a decaying field: $h_u = h_0/|u|^\delta$. Finally, we have the case we will be interested in this text: the random field, which will be extensively studied in the next section. Before, however, we will present a brief summary of the discoveries about the usual Ising Model.

The model was first considered in the 1920s by Lenz and Ising, which proved that this model does not undergoes a phase transition in one-dimension. In 1936, Peierls proved the existence of phase transition for dimension greater than one $h \equiv 0$. In 1941, Kramers and Wannier managed to find the critical temperature for the two-dimensional model, which was later confirmed by Onsager who deduced the analytical solution for the two-dimensional model with absence of field in 1944. Due to contributions by Dyson (1969) and Frölich-Spencer (1982), it was proved that there is phase transition in the long-range 1D model if and only if the exponent $\alpha \in (1, 2]$.

[4] Seeing it another (and very interesting) way, we are studying countable families of random variables that are dependent with one another.

3 Disordered Systems

In the typical Ising model, the parameters J_{ij} and h_i are fixed, but it is worthwhile to consider them to be random variables. One motivation for this is that real solids rarely have a perfect lattice structure — the sites may be scruffy, some site may be missing or there may be more than one element, such as in alloys. Consider an alloy consisting of atoms of iron and gold, for example. We have no hope to be able to tell the exact distribution of the two types of atoms throughout the solid, so we assume that each distribution may give rise to a different family of parameters J_{ij} and h_i. In this case, the hamiltonian will be also a random variable. The hamiltonian for Ising-type models, for example, will become:

$$H^{\eta}_{\Lambda}[\omega](\sigma) = -\sum_{i,j\in\Lambda} J_{ij}[\omega]\sigma_i\sigma_j - \sum_{i\in\Lambda,\ j\notin\Lambda} J_{ij}[\omega]\sigma_i\eta_j - \sum_{i\in\Lambda} h_i[\omega]\sigma_i$$

Where each ω is a specific realization. The case where J_{ij} indeed depends on ω is usually called "spin glass", but in this text we will be concerned on the case where J_{ij} is fixed (and constant, as in the usual model) in such a way that the source of randomness will be only the external field. This model is called Random Field Ising Model, or RFIM.

A little bit of history about this model. In 1975, Imry and Ma [5] argued that this model has phase transition if, and only if $d \geq 3$. This argument, however, became controversial when Parisi and Sourlas [8] used field theory techniques in 1979 to argue that RFIM in dimension d should behave as the Ising model in dimension $d-2$, the so-called dimension reduction argument, so that the $3d$ RFIM would not have phase transition. This controversy was finally overwhelmed by Bricmont and Kupiainen [3] in 1988, showing that Imry and Ma were right, through the usage of renormalization group theory. Completing the picture, it was the turn of Aizenman and Wehr [2] to show, in 1990, that Imry-Ma was also correct about the lack of transition in dimension 2 or below.

Reviewing [4], the goal of this essay is to obtain the same results of [3], but without having to appeal to renormalization group techniques and in such a way to naturally extend the results to more complex models, such as the Potts model, which is not so feasible with the former method. This will be accomplished by a variation of the Peierls argument, which will be done in the joint of the configuration space and the space of randomness associated to the external field.

3.1 Disorder Space

Before going on, we must put the random effects in a more rigorous framework. It may be modeled by a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ and a family $(h_u)_{u\in\mathbb{Z}^d}$ of i.i.d (independent and identically distributed) real gaussian random variables defined in Ω. This way, H_Λ also becomes a random variable (already defined above), as well as the partition functions $Z^{\eta}_{\Lambda,\beta}$ and the Gibbs measures $\mu^{\eta}_{\Lambda,\beta}$.

Explicitly, we have (supposing uniform a priori measure):

$$Z^{\eta}_{\Lambda,\beta}[\omega] = \sum_{\sigma_\Lambda\in\mathcal{S}_\Lambda} e^{-\beta H^{\eta}_{\Lambda}[\omega](\sigma_\Lambda\eta_{\Lambda^c})}$$

$$\mu^{\eta}_{\Lambda,\beta}[\omega](\sigma) = \frac{e^{-\beta H^{\eta}_{\Lambda}[\omega](\sigma)}}{Z^{\eta}_{\Lambda,\beta}[\omega]}$$

Notice also that there will be a kind of identification between ω and the family (h_u), so we may write the quantities above changing the dependence on ω by a dependence on h:

$$H^{\eta}_{\Lambda,h}(\sigma_\Lambda) = -\sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - \sum_{i\in\Lambda,\ j\notin\Lambda} J_{ij}\sigma_i\eta_j - \sum_{i\in\Lambda} h_i\sigma_i$$

$$Z^{\eta}_{\Lambda,\beta}(h) = \sum_{\sigma_\Lambda\in\mathcal{S}_\Lambda} e^{-\beta H^{\eta}_{\Lambda,h}(\sigma_\Lambda)}$$

$$\mu^{\eta}_{\Lambda,\beta,h}(\sigma_\Lambda) = \frac{e^{-\beta H^{\eta}_{\Lambda,h}(\sigma)}}{Z^{\eta}_{\Lambda,\beta}(h)}$$

To make things more concrete, we will present the basic idea for the construction of $(\Omega, \mathcal{A}, \mathbb{P})$, which will in turn also justify the identification aforementioned. We may take $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ and $\mathcal{A} = \mathcal{B}^{\mathbb{Z}^d}$ the product $\sigma-$algebra of $\mathcal{B}$ that is the Borel $\sigma-$algebra of $\mathbb{R}$. The most difficult part is precisely defining a suitable measure $\mathbb{P}$. For that, we may specify the probability in the class of the union of rectangular cylinders with finite base, which forms an algebra and use some extension theorem, like the Caratheodory theorem. This is the idea behind the so-called Ionescu-Tulcea theorem, which is the standard way of constructing measures in a countably infinite product of spaces. For our needs, it will suffice the following theorem, which is actually a corollary of the Ionescu-Tulcea.

Theorem 3.1: Countable Product of Measures: [6]

Let I be a countable set and let $(\Omega_i, \mathcal{A}_i, P_i)$ be a family of probability spaces indexed by I. Then there exists a uniquely determined probability measure $\mathbb{P}$ on the product space $(\Omega, \mathcal{A})$ which satisfies:

$$\mathbb{P}\left(A_0 \times ... \times A_n \times \bigtimes_{i=n+1}^{\infty} \Omega_i\right) = \prod_{k=0}^{n} P_k(A_k) \tag{1}$$

For $A_i \in \mathcal{A}_i$, $i = 0, ..., n$ and $n \in \mathbb{N}_0$.
$\mathbb{P}$ is called the product of the measures (P_i). The coordinate maps (X_i) are independent under $\mathbb{P}$.

For obvious reasons, we will call the coordinate maps h_i instead of X_i. Let $\mathcal{N}_{0,1}$ be the measure in $\mathbb{R}$ such that:

$$\mathcal{N}_{0,1}(A) = \int \chi_A(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}d\lambda(x)$$

Where λ is the Lebesgue measure. That is, $\mathcal{N}_{0,1}$ is normally distributed.

Then, with the aid of the previous theorem, we define $\mathbb{P}$ in $(\mathbb{R}, \mathcal{B})^{\otimes\mathbb{Z}^d} := (\mathbb{R}^{\mathbb{Z}^d}, \mathcal{B}^{\mathbb{Z}^d})$ with $P_i = \mathcal{N}_{0,1}$ for every $i \in \mathbb{Z}^d$. In particular, for $A \in \mathcal{B}$:

$$\mathbb{P}(h_i \in A) := \mathbb{P}(h_i^{-1}(A)) = \mathbb{P}\left(\mathbb{R} \times ... \times A \times \bigtimes_{j=i+1}^{\infty} \mathbb{R}\right) = \prod_{k=0}^{n-1} \mathcal{N}_{0,1}(\mathbb{R}) \cdot \mathcal{N}_{0,1}(A) = \mathcal{N}_{0,1}(A)$$

So the random variables $\{h_i\}$ are distributed exactly as we wished them to be.

The random variables (h_u) are frequently supposed to be "normalized", in the sense that they all have variance 1. In order to control the strength of the field (i.e., its variance), one may multiply each random variable by ϵ. Thus, we are going to be actually concerned about the random variables $(\epsilon h_u)_{u \in \mathbb{Z}^d}$. If h_u is gaussian distributed with variance 1, then ϵh_u is normally distributed with variance ϵ^2. This is easily seen in intervals:

$$\mathbb{P}(\epsilon h_u \in [a, b)) = \mathbb{P}(h_u \in [a/\epsilon, b/\epsilon)) = \frac{1}{\sqrt{2\pi}} \int_{a/\epsilon}^{b/\epsilon} e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi\epsilon^2}} \int_a^b e^{-\frac{u^2}{2\epsilon^2}} du$$

Where we made the simple change of variables $u = \epsilon x$. Notice that it is defined in the semi-ring of intervals and we may use standard extension theorems to extend it to $\mathcal{B}$. The uniqueness guarantees that this is really the gaussian distribution we were looking for. (We could simply integrate on an arbitrary borel set and procedure similarly with the change of variables).

4 Long Range Order

For the subsequent discussion, $\Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$ is the box of side $2n$ centered at 0.

Usually, long range order means that the correlation function of two spins have a positive lower bound. In [4], however, it is defined with the aid of a quantity that will be called here as "order parameter". This quantity has, indeed, something to do with correlation functions, as showed in [1] and will be presented in the next subsection.

4.1 Order parameter

Coming soon.

4.2 Quenched Measure

Fix $N > 0$. We will start by defining a measure in $(\mathbb{R}^{\Lambda_N} \times \mathcal{S}_{\Lambda_N}, \mathcal{B}(\mathbb{R}^{\Lambda_N}) \otimes \mathcal{F}_{\Lambda_N})$,[5] remembering that $\mathcal{S}_\Lambda := (\mathcal{S}_0)^\Lambda = \{-1, 1\}^\Lambda$. For A a borelian from $\mathbb{R}^{\Lambda_N}$ and $B \subset \mathcal{S}_{\Lambda_N}$, we define:

$$\mathbb{Q}^\eta_{\beta,\Lambda_N,\epsilon}(A \times B) = \int_A \mu^\eta_{\beta,\Lambda_N,\epsilon h}(B)\, d\mathbb{P}(h)$$

Notice that we are fixing some $h \in A$, calculating $\mu^\eta_{\beta,\Lambda_N,\epsilon h}(B)$ (which depends on h), and then integrating over A, so it is way we called this measure the "quenched measure". Writing the integral in a more explicit way, we will first expand μ. As the a priori measure is uniform, we have:

$$\mu^\eta_{\Lambda,\beta,\epsilon h}(\sigma) = \frac{e^{-\beta H^\eta_{\Lambda,\epsilon h}(\sigma)}}{Z^\eta_{\Lambda,\beta,\epsilon}(h)} \implies \mu^\eta_{\Lambda,\beta,h}(B) = \sum_{\sigma \in B} \frac{e^{-\beta H^\eta_{\Lambda,\epsilon h}(\sigma)}}{Z^\eta_{\Lambda,\beta,\epsilon}(h)}$$

So:

$$\mathbb{Q}^\eta_{\beta,\Lambda_N,\epsilon}(A \times B) = \int_A \frac{1}{Z^\eta_{\Lambda_N,\beta,\epsilon}(h)} \sum_{\sigma \in B} \exp\bigl(-\beta H^\eta_{\Lambda_N,\epsilon h}(\sigma)\bigr)\, d\mathbb{P}$$

And using the definition of the fact that h are Gaussian distributed we can integrate with respect to the usual Lebesgue measure:

$$\mathbb{Q}^\eta_{\beta,\Lambda_N,\epsilon}(A \times B) = \int_A \frac{1}{Z^\eta_{\Lambda_N,\beta,\epsilon}(h)} \sum_{\sigma \in B} \exp\bigl(-\beta H^\eta_{\Lambda_N,\epsilon h}(\sigma)\bigr) \times \prod_{u \in \Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{h_u^2}{2}} (dh_u)_{u \in \Lambda_N}$$

[5]This measure could also be defined in the whole product $\mathbb{R}^{\mathbb{Z}^d} \times \mathcal{S}$ as long as the $\sigma-$algebra of $\mathbb{R}^{\mathbb{Z}^d}$ which participates in the product $\sigma-$algebra is generated by cylinders supported on Λ_N. See the discussion of such identifications at the end of the subsection 1.2.2.

$$\int_A \sum_{\sigma\in B} \nu^\eta_{\Lambda_N,\beta,\epsilon}(h,\sigma)(dh_u)_{u\in\Lambda_N} \overset{\text{fub.}}{=} \int_{A\times B} \nu^\eta_{\Lambda_N,\beta,\epsilon}(h,\sigma)\; d\lambda(h)\otimes d\#(\sigma)$$

Where λ and $\#$ are respectively the Lebesgue and counting mesasure and we have defined:

!

$$\nu^\eta_{\Lambda_N,\beta,\epsilon}(h,\sigma) := \frac{\exp\big(-\beta H^\eta_{\Lambda_N,\epsilon h}(\sigma)\big)}{Z^\eta_{\Lambda_N,\beta,\epsilon}(h)} \times \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{h_u^2}{2}}$$

Proposition 4.1

$\mathbb{Q}$ is indeed a measure

Proof. Coming soon. ■

Now, we are going to deduce some intuitive and useful identities to use later. In first place, notice that:

$$\mathbb{Q}^\eta(A\times\mathcal{S}_{\Lambda_N}) = \int_A \frac{1}{Z^\eta_{\Lambda_N,\beta,\epsilon}(h)} \sum_{\sigma\in\mathcal{S}_{\Lambda_N}} \exp\big(-\beta H^\eta_{\Lambda_N,\epsilon h}(\sigma)\big) \times \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{h_u^2}{2}} (dh_u)_{u\in\Lambda_N}$$

$$= \int_A \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{h_u^2}{2}} (dh_u)_{u\in\Lambda_N} = \int_{\mathbb{R}^{\Lambda_N}} \chi_A(h) \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{h_u^2}{2}} (dh_u)_{u\in\Lambda_N} = \int_{\mathbb{R}^{\Lambda_N}} \chi_A(h) d\mathbb{P}(h)$$

Where we used the definition of the partition function in the second equality. Thus, we conclude, as expected, that:

$$\mathbb{Q}^\eta(A\times\mathcal{S}_{\Lambda_N}) = \mathbb{P}(A) \tag{2}$$

4.3 Peierls Argument

In this section we will fix the boundary condition to be the positive.

Definition 4.1

We denote by $\Gamma_o^N = \Gamma_o$ the set of simply connected subsets of Λ_N containing 0. Given $\sigma\in\mathcal{S}_{\Lambda_N}$, we denote by $\mathcal{A}_\sigma$ the smaller simply connected component containing the origin and its sign component. In other words, $\mathcal{A}_\sigma$ is the simply connected component enclosed by the outmost boundary of the sign component at the origin.

For $A\subset\Lambda$, we define σ^A by the configuration in $\mathcal{S}_\Lambda$ such that $\sigma_i^A = \sigma_i i$ if $i\notin A$ and $\sigma_i^A = -\sigma_i$ if $i\in A$. We define h^A analogously. It is easily seen that, for every η there is one and exactly one η such that $\eta = \sigma^A$, and so this is a bijective map of $\mathcal{S}_\Lambda$ onto itself.

With this in mind, we have:

$$\nu^+_{\Lambda_N,\beta,\epsilon}(\sigma^A, h^A) = \frac{1}{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)} \exp\Big(-\beta H^+_{\Lambda_N,\epsilon h^A}(\sigma^A)\Big) \times \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{(h_u^A)^2}{2}}$$

The most obvious is that the gaussian distribution does not chance with the transformation $h \to h^A$ be cause the field is squared in there, so the sign does not matter. By the fact that this change is a bijection, if we sum up all the configurations, we will have the same result, that is:

$$\sum_\sigma \exp\Big(-\beta H^+_{\Lambda_N,\epsilon h^A}(\sigma^A)\Big) = \sum_\sigma \exp\Big(-\beta H^+_{\Lambda_N,\epsilon h^A}(\sigma)\Big) = Z^+_{\Lambda_N,\beta,\epsilon}(h^A)$$

$$\implies \sum_\sigma \nu^+_{\Lambda_N,\beta,\epsilon}(\sigma^A, h^A) = \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{(h_u^A)^2}{2}} = \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{(h_u)^2}{2}}$$

$$\implies \int_A \sum_\sigma \nu^+_{\Lambda_N,\beta,\epsilon}(h^A, \sigma^A) dh = 1 \tag{3}$$

For the general case, where we not necessarily sum up all the configurations, let's look how the hamiltonian changes for an specific configuration:

$$H^+_{\Lambda_N,\epsilon h^A}(\sigma^A)) = -\sum_{i\sim j\in\Lambda_N\setminus A} J\sigma_i\sigma_j - \sum_{i\sim j\in A} J(-\sigma_i)(-\sigma_j) - \sum_{i\sim j;\ i\in A,\ j\in\Lambda_N\setminus A} J(-\sigma_i)(\sigma_j)$$

$$-\sum_{i\sim j\ i\in\Lambda_N\setminus A,\ j\notin\Lambda_N} J\sigma_i - \sum_{i\sim j;\ i\in A,\ j\notin\Lambda_N} J(-\sigma_i) - \sum_{i\in\Lambda_N\setminus A} \epsilon h_i\sigma_i - \sum_{i\in A} \epsilon(-h_i)(-\sigma_i)$$

Reorganizing the terms and trying to write in the terms of the original hamiltonian leaves us with:

$$H^+_{\Lambda_N,\epsilon h^A}(\sigma^A)) = H^+_{\Lambda_N,\beta,\epsilon h}(\sigma) + 2\sum_{i\sim j;\ i\in A,\ j\in\Lambda_N\setminus A} J\sigma_i\sigma_j + 2\sum_{i\sim j;\ i\in A,\ j\notin\Lambda_N} J\sigma_i\sigma_j$$

For the Peierls argument we usually make the assumption that if $i \sim j$ with $i \in A$ and $j \in A^c$, then $\sigma_i\sigma_j = -1$, so:

$$H^+_{\Lambda_N,\epsilon h^A}(\sigma^A)) = H^+_{\Lambda_N,\beta,\epsilon h}(\sigma) - \sum_{i\sim j;\ i\in A,\ j\in\Lambda_N\setminus A} 2J - \sum_{i\sim j;\ i\in A,\ j\notin\Lambda_N} 2J$$

Finally, notice that we are summing $2J$ over all pairs of points such that $i \in A$ and $j \notin A$, which is exactly the elements of ∂A, and we get:

$$H^+_{\Lambda_N,\epsilon h^A}(\sigma^A)) = H^+_{\Lambda_N,\beta,\epsilon h}(\sigma) - 2J|\partial A|$$

So:

$$\nu^+_{\Lambda_N,\beta,\epsilon}(\sigma^A, h^A) = \frac{1}{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)} \exp\big(-\beta H^+_{\Lambda_N,\beta,\epsilon h}(\sigma) + 2\beta J|\partial A|\big) \times \prod_{u\in\Lambda_N} \frac{1}{\sqrt{2\pi}} e^{-\frac{h_u^2}{2}}$$

! $$\Longrightarrow \frac{\nu^+_{\Lambda_N,\beta,\epsilon}(\sigma, h)}{\nu^+_{\Lambda_N,\beta,\epsilon}(\sigma^A, h^A)} = \frac{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)}{Z^+_{\Lambda_N,\beta,\epsilon}(h)} e^{-2\beta J|\partial A|} \tag{4}$$

We will need to bound the quotient of the partition function, which depends on the external field. Indeed, the probability that this quotient is large is very small, as the next subsection will show.

4.4 Interlude: A bound for Extreme Events

In order to study the quotient of the partition functions, we define:

! $$\Delta_A(h) = \frac{1}{\beta J}\big(\ln Z^+_{\Lambda_N,\beta,\epsilon}(h^A) - \ln Z^+_{\Lambda_N,\beta,\epsilon}(h)\big)$$

Define:

$$\mathcal{E} := \left\{ h;\ \frac{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)}{Z^+_{\Lambda_N,\beta,\epsilon}(h)} \leq e^{\beta J|\partial A|},\ \forall A \in \Gamma_o \right\}$$

Intuitively, $\mathcal{E}$ is the set of fields that makes the quotient of the partitions functions do not be so big. We have:

$$h \notin \mathcal{E} \iff \frac{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)}{Z^+_{\Lambda_N,\beta,\epsilon}(h)} > e^{\beta J|\partial A|},\ \text{for some } A \in \Gamma_o$$

But:

$$\frac{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)}{Z^+_{\Lambda_N,\beta,\epsilon}(h)} = \exp\big(\ln Z^+_{\Lambda_N,\beta,\epsilon}(h^A) - \ln Z^+_{\Lambda_N,\beta,\epsilon}(h)\big) = \exp(\beta J \Delta_A(h))$$

So:

$$h \notin \mathcal{E} \iff e^{\beta J \Delta_A(h)} > e^{\beta J|\partial A|},\ \text{for some } A \in \Gamma_o$$

$$\iff \Delta_A(h) > |\partial A| \iff \frac{\Delta_A(h)}{|\partial A|} > 1,\ \text{for some } A \in \Gamma_o$$

$$\iff \sup_{A\in\Gamma_o} \frac{\Delta_A(h)}{|\partial A|} > 1$$

And we conclude that:

$$\mathcal{E}^c = \left\{h;\ \sup_{A\in\Gamma_o} \frac{\Delta_A(h)}{|\partial A|} > 1\right\}$$

Lemma 4.1

For $d \geq 3$, it is true that:

$$\mathbb{P}(\mathcal{E}^c) \leq \exp\bigl(-c/\epsilon^2\bigr)$$

4.5 Peierls Argument - Continuation

The previous section will imply that a singular configuration must not be "so disordered", and now we have to bound the number of such configurations. This will be our next goal.

Lemma 4.2

$$\#\{A \in \Gamma_0;\ |\partial A| = n\} \leq (2dn)^d (16d^3)^{2n}$$

Proof. ■

All that was done until now allows us to prove the following:

Lemma 4.3

There is $c > 0$ such that:

$$\mathbb{Q}^+(\mathbb{R}^{\Lambda_N} \times \sigma_0^{-1}(-1)) \leq e^{-\beta Jc} + e^{-c/\epsilon^2}$$

For β large enough.

Proof. We have:

$$\mathbb{Q}^+((\mathcal{E} \cup \mathcal{E}^c) \times \sigma_0^{-1}(-1)) = \mathbb{Q}^+(\mathcal{E} \times (\sigma_0^{-1}(-1)) \cup (\mathcal{E}^c \times \sigma_0^{-1}(-1)))$$

$$= \mathbb{Q}^+(\mathcal{E} \times \sigma_0^{-1}(-1)) + \mathbb{Q}^+(\mathcal{E}^c \times \sigma_0^{-1}(-1)) \leq \mathbb{Q}^+(\mathcal{E} \times \sigma_0^{-1}(-1)) + \mathbb{Q}^+(\mathcal{E}^c \times \mathcal{S}_{\Lambda_N})$$

By (2), we conclude that $\mathbb{Q}^+(\mathcal{E}^c \times \mathcal{S}_{\Lambda_N}) = \mathbb{P}(\mathcal{E}^c)$. And thus: $\mathbb{Q}^+(\mathbb{R}^{\Lambda_N} \times \sigma_0^{-1}(-1)) \leq \mathbb{Q}^+(\mathcal{E} \times \sigma_0^{-1}(-1)) + \mathbb{P}(\mathcal{E}^c)$. As for the first term, in first place, we remark that the set of configurations $\sigma \in \mathcal{S}_{\Lambda_N}$ such that $\sigma_0 = -1$ is a subset of the configurations such that $\mathcal{A}_\sigma \in \Gamma_o$, and hence:

$$\mathbb{Q}^+(\mathcal{E}\times\sigma_0^{-1}(-1)) = \int_{\mathcal{E}}\sum_{\sigma;\ \sigma_0=-1}\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)(dh_u)_{u\in\Lambda_N} \leq \sum_{A\in\Gamma_0}\int_{\mathcal{E}}\sum_{\sigma;\ \mathcal{A}_\sigma=A}\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)(dh_u)_{u\in\Lambda_N}$$

By (3), we have:

$$\int_{\mathcal{E}}\sum_{\sigma;\mathcal{A}_\sigma=A}\nu^+_{\Lambda_N,\beta,\epsilon}(h^A,\sigma^A)dh \leq \int\sum_{\sigma}\nu^+_{\Lambda_N,\beta,\epsilon}(h^A,\sigma^A)dh = 1$$

$$\implies \mathbb{Q}^+(\mathcal{E}\times\sigma_0^{-1}(-1)) \leq \sum_{A\in\Gamma_0}\frac{\int_{\mathcal{E}}\sum_{\sigma;\ \mathcal{A}_\sigma=A}\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)dh}{\int_{\mathcal{E}}\sum_{\sigma;\mathcal{A}_\sigma=A}\nu^+_{\Lambda_N,\beta,\epsilon}(h^A,\sigma^A)dh}$$

Focusing for a moment on the numerator:

$$\int_{\mathcal{E}}\sum_{\sigma;\ \mathcal{A}_\sigma=A}\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)dh = \int_{\mathcal{E}}\sum_{\sigma;\ \mathcal{A}_\sigma=A}\frac{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)}{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma}\nu^+_{\Lambda_N,\beta,\epsilon}(h^A,\sigma^A)dh$$

$$\leq \int_{\mathcal{E}}\sum_{\sigma;\ \mathcal{A}_\sigma=A}\sup_{h\in\mathcal{E},\ \sigma;\mathcal{A}_\sigma=A}\left(\frac{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)}{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma}\right)\nu^+_{\Lambda_N,\beta,\epsilon}(h^A,\sigma^A)dh$$

$$= \sup_{h\in\mathcal{E},\ \sigma;\mathcal{A}_\sigma=A}\left(\frac{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)}{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma}\right)\int_{\mathcal{E}}\sum_{\sigma;\ \mathcal{A}_\sigma=A}\nu^+_{\Lambda_N,\beta,\epsilon}(h^A,\sigma^A)dh$$

The integral cancels out with the denominator and we are left with:

$$\mathbb{Q}^+(\mathcal{E}\times\sigma_0^{-1}(-1)) \leq \sum_{A\in\Gamma_0}\sup_{h\in\mathcal{E},\ \sigma;\mathcal{A}_\sigma=A}\left(\frac{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma)}{\nu^+_{\Lambda_N,\beta,\epsilon}(h,\sigma}\right) \overset{(4)}{=} \sum_{A\in\Gamma_0}\sup_{h\in\mathcal{E}}\left(\frac{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)}{Z^+_{\Lambda_N,\beta,\epsilon}(h)}e^{-2\beta J|\partial A|}\right)$$

Now we use the definition of $\mathcal{E}$ and separate the summation to get:

$$\mathbb{Q}^+(\mathcal{E}\times\sigma_0^{-1}(-1)) \leq \sum_{n\leq 1}\sum_{A\in\Gamma_0,|\partial A|=n}e^{-2\beta J|\partial A|}\sup_{h\in\mathcal{E}}\left(\frac{Z^+_{\Lambda_N,\beta,\epsilon}(h^A)}{Z^+_{\Lambda_N,\beta,\epsilon}(h)}\right) \leq \sum_{n\geq 1}\sum_{A\in\Gamma_0,|\partial A|=n}e^{-2\beta J|\partial A|+\beta J|\partial A|}$$

$$\mathbb{Q}^+(\mathcal{E}\times\sigma_0^{-1}(-1)) \leq \sum_{n\geq 1}\#\{A\in\Gamma_0;\ |\partial A|=n\}e^{-\beta Jn}$$

Invoking **??**, we are left with:

$$\mathbb{Q}^+(\mathcal{E} \times \sigma_0^{-1}(-1)) \leq \sum_{n\geq 1}(2dn)^d(16d^3)^{2n}e^{-\beta Jn}$$

So now we are going to prove that there exists $\beta > 0$ which makes this series be less than $e^{-\beta Jc}$. The idea is to make each term less than $2^{-n}e^{-\beta Jc}$, and the conclusion will follow. The detail of the multiplicative factor of the exponential is not important, so we are going to write it as $An^\gamma B^n$, with A, B, γ constants. To simplify, we will divide both sides by $2^{-n}e^{-\beta Jc}$, so we will need to find a c such that:

$$An^\gamma B^n$$

■

Theorem 4.1: [4]

For $d \geq 3$, there exists a constant $c > 0$ such that, for all $0 \leq T, \epsilon \leq c$ and all $N \geq 1$, we have:

$$\mathbb{P}(\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1)) \leq e^{-c\beta J} + e^{-c/\epsilon^2}) \geq 1 - e^{-c\beta J} - e^{-c/\epsilon^2}$$

Proof. Markov inequality says that:

$$\mathbb{P}(\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1) \geq a) \leq \frac{\mathbb{E}(\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1))}{a} = \frac{\mathbb{Q}(\sigma_0 = -1)}{a}$$

So, **??** says there is a constant c_1 such that:

$$\mathbb{P}(\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1) \geq a) \leq \frac{e^{-c_1\beta J} + e^{-c_1/\epsilon^2}}{a}$$

$$\iff \mathbb{P}(\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1) \leq a) \geq 1 - \frac{e^{-c_1\beta J} + e^{-c_1/\epsilon^2}}{a}$$

Now, let $c = c_1/2$. Then we get:

$$\iff \mathbb{P}(\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1) \leq e^{-c\beta J} + e^{-c/\epsilon^2}) \geq 1 - \frac{e^{-c_1\beta J} + e^{-c_1/\epsilon^2}}{e^{-c\beta J} + e^{-c/\epsilon^2}}$$

But notice that:

$$(e^{-c\beta J} + e^{-c/\epsilon^2})^2 = e^{-c_1\beta J} + e^{-c_1/\epsilon^2} + 2e^{-c_1\beta J - c_1/\epsilon^2} \geq e^{-c_1\beta J} + e^{-c_1/\epsilon^2}$$

$$\implies -\frac{e^{-c_1\beta J} + e^{-c_1/\epsilon^2}}{e^{-c\beta J} + e^{-c/\epsilon^2}} \geq -e^{-c\beta J} - e^{-c/\epsilon^2}$$

And then:

$$\mathbb{P}(\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0=-1)\leq e^{-c\beta J}+e^{-c/\epsilon^2})\geq 1-e^{-c\beta J}-e^{-c/\epsilon^2}$$

As we wanted.

■

4.6 Ergodicity

We did not prove the existence of phase transition yet. What we really want to prove is that the set of fields which makes μ_h^+ and μ_h^- differ has probability one. We will present some important definitions we must deal with in order to prove so.

Definition 4.2

Let $(\Omega,\mathcal{A},\mathbb{P},\tau)$ be a probability space with $\tau:\Omega\to\Omega$ a measurable function (i.e. a dynamical system). A set $A\in\mathcal{A}$ is called invariant if $\tau^{-1}(A)=A$. $\mathbb{P}$ is called invariant if $\mathbb{P}(\tau^{-1}(A))=\mathbb{P}(A)$ for every $A\in\mathcal{A}$.

We say that $(\Omega,\mathcal{A},\mathbb{P},\tau)$ is ergodic if $\mathbb{P}$ is invariant and $\mathbb{P}(A)=0$ or $\mathbb{P}(A)=1$ for every A invariant.

We say that $(\Omega,\mathcal{A},\mathbb{P},\tau)$ is (strongly) mixing if $\mathbb{P}$ is invariant and:

$$\lim_{n\to\infty}\mathbb{P}(A\cap\tau^{-n}(B))=P(A)P(B) \tag{5}$$

for every $A,B\in\mathcal{A}$

It is a classical result that strongly mixing dynamical systems are ergodic, so the idea is to prove that our system is mixing and then prove that the set of interest is invariant. We will start by showing that $\mathbb{P}$ is invariant.

But what will be our dynamic τ?. We define $\tau(\omega)_i=\omega_{i-e_1}$, where $e_1=(1,0,...,0)$

We notice that $\mathbb{P}$ is invariant for the cylinder sets. Indeed, let $\mathscr{C}_j(A):=\bigtimes_{k\in\mathbb{Z}^d}A_k$, where $A_k=\mathbb{R}$ if $k\neq j$ and $A_j=A$. It is straightforward that $\tau(\mathscr{C}_j(A))=\mathscr{C}_{j+e_1}(A)$ and $\tau^{-1}(\mathscr{C}_j(A))=\mathscr{C}_{j-e_1}(A)$. By the construction of $\mathbb{P}$ and the previous remark we easily have that $\mathbb{P}(\tau^{-1}(\mathscr{C}_j(A)))=\mathbb{P}(\mathscr{C}_j(A))$ for every $j\in\mathbb{N}$ and $A\subset\mathcal{B}$.

The next step would naturally be to prove that the class of all sets which makes $\mathbb{P}$ invariant forms a $\sigma-$algebra, and it would follow that every measurable makes $\mathbb{P}$ invariant. This is not the case, though. It is not difficult to see that the class of such sets is closed by complements, by disjoint union and by difference of sets such that $A\subset B$.

Consider the following example. Take $\Omega=\{a,b,c,d\}$, with the $\sigma-$algebra of the parts and such that $\mu(a)=\mu(c)\neq\mu(b)=\mu(d)$. Let T be defined by $T(a)=b,T(b)=a,T(c)=d,T(d)=c$. If $A=\{a,b\}$ and $B=\{b,c\}$, then A and B are invariant, so in particular they make μ invariant. However, $A\cap B=\{b\}$ which clearly doesn't make μ invariant. The situation is bad even with reasonable hypothesis, just notice that T is a bijection and that the class of sets that makes μ

invariant generates the $\sigma-$algebra. This fact hampers the proof that $\mathbb{P}$ is ergodic in our case, so we will need another strategy. In order to better tackle this problem, we will make use of the Dynkin $\pi - \lambda$ theorem:

Definition 4.3: $\lambda-$system

A class of subsets $\mathcal{D}$ of Ω is called a $\lambda-$system for Ω if: (i) $\Omega \in \mathcal{D}$, (ii) $\mathcal{D}$ is closed under complements and (iii) $\mathbb{D}$ is closed under countable union of disjoint sets.
For some $\mathcal{X} \subset 2^{\Omega}$, $\delta(\mathcal{X})$ is defined as the minimal λ system that contains $\mathcal{D}$, and called the $\lambda-$systems generated by $\mathcal{X}$.

Theorem 4.2: Dynkin $\pi - \lambda$ theorem [6]

Let $\mathcal{X} \subset 2^{\Omega}$ be closed under intersections (i.e., a $\pi-$systems). then:

$$\sigma(\mathcal{X}) = \delta(\mathcal{X})$$

In particular, if $\mathcal{D}$ is a $\lambda-$system that contains $\mathcal{X}$, then $\sigma(\mathcal{X}) \subset \mathcal{D}$

Notice that the lacking hypothesis in our counterexample is the fact that the class of sets which makes the measure invariant was not closed under intersection, but fortunately this is the case for the cylinder sets. Recall that the intersection of two cylinders with finite base is also a cylinder with finite base, and it is straightforward to prove that every such cylinder makes $\mathbb{P}$ invariant.

Proposition 4.2

The dynamical system $(\Omega, \mathcal{A}, \mathbb{P}, \tau)$ is strongly mixing.

Proof. We already proven that $\mathbb{P}$ is invariant. Now, let B be an arbitrary cylinder and let $\mathcal{A}_B$ be the collection of sets such that:

$$\lim_{n\to\infty} \mathbb{P}(A \cap \tau^{-n}(B)) = \mathbb{P}(A)\mathbb{P}(B)$$

I will show that $\mathcal{A}_B$ is a $\lambda-$system.

(a) $\Omega \in \mathcal{A}_B$

$$\lim_{n\to\infty} \mathbb{P}(\Omega \cap \tau^{-n}(B)) = \lim_{n\to\infty} \mathbb{P}(\tau^{-n}(B)) = \lim_{n\to\infty} \mathbb{P}(B) = \mathbb{P}(B) = \mathbb{P}(\Omega)\mathbb{P}(B)$$

(b) $A \in \mathcal{A}_B \implies A^c \in \mathcal{A}_B$

$$\lim_{n\to\infty} \mathbb{P}(A^c \cap \tau^{-n}(B)) = \lim_{n\to\infty} \mathbb{P}(\tau^{-n}(B) \backslash A) = \lim_{n\to\infty} \mathbb{P}(\tau^{-n}(B)) - \mathbb{P}(A \cap \tau^{-n}(B))$$

$$= \mathbb{P}(B) - \lim_{n\to\infty} \mathbb{P}(A \cap \tau^{-n}(B)) = \mathbb{P}(B) - \mathbb{P}(A)\mathbb{P}(B) = (1 - \mathbb{P}(A))\mathbb{P}(B) = \mathbb{P}(A^c)\mathbb{P}(B)$$

(c) If $(A_k)_{k\geq 0}$ is a pairwise disjoint sequence of sets such that $A_k \in \mathcal{A}_B$ for every k, then $A := \bigcup_{k\geq 0} A_k \in \mathcal{A}_B$

Defining $A^N := \bigcup_{k>N} A_k$, then:

$$\left|\mathbb{P}(A\cap\tau^{-n}(B)) - \mathbb{P}(A)\mathbb{P}(B)\right| = \left|\mathbb{P}\left(\left(\bigcup_{k=0}^{N} A_k \cup A^N\right)\cap\tau^{-n}(B)\right) - \mathbb{P}\left(\bigcup_{k=0}^{N} A_k \cup A^N\right)\mathbb{P}(B)\right|$$

$$= \left|\mathbb{P}\left(\left(\bigcup_{k=0}^{N} A_k \cap\tau^{-n}(B)\right)\cup\left(A^N\cap\tau^{-n}(B)\right)\right) - \left[\mathbb{P}\left(\bigcup_{k=0}^{N} A_k\right) + \mathbb{P}(A^N)\right]\mathbb{P}(B)\right|$$

$$= \left|\mathbb{P}\left(\bigcup_{k=0}^{N} A_k \cap\tau^{-n}(B)\right) - \mathbb{P}\left(\bigcup_{k=0}^{N} A_k\right)\mathbb{P}(B) + \mathbb{P}\left(A^N\cap\tau^{-n}(B)\right) - \mathbb{P}(A^N)\mathbb{P}(B)\right|$$

$$\leq \left|\mathbb{P}\left(\bigcup_{k=0}^{N} A_k \cap\tau^{-n}(B)\right) - \mathbb{P}\left(\bigcup_{k=0}^{N} A_k\right)\mathbb{P}(B)\right| + \left|\mathbb{P}\left(A^N\cap\tau^{-n}(B)\right) - \mathbb{P}(A^N)\mathbb{P}(B)\right|$$

$$\leq \left|\mathbb{P}\left(\bigcup_{k=0}^{N} A_k \cap\tau^{-n}(B)\right) - \mathbb{P}\left(\bigcup_{k=0}^{N} A_k\right)\mathbb{P}(B)\right| + \mathbb{P}\left(A^N\cap\tau^{-n}(B)\right) + \mathbb{P}(A^N)\mathbb{P}(B)$$

Now, given $\epsilon > 0$, we take N such that $\mathbb{P}(A^N) < \epsilon/3$, and this exists because $\lim_{N\to\infty}\mathbb{P}(A^N) = 0$ by the fact that $\bigcap_{N\geq 0} A^N = \emptyset$ and the semicontinuity. Now, we remark that, for n sufficiently large, we have:

$$\left|\mathbb{P}\left(\bigcup_{k=0}^{N} A_k \cap\tau^{-n}(B)\right) - \mathbb{P}\left(\bigcup_{k=0}^{N} A_k\right)\mathbb{P}(B)\right| < \epsilon/3$$

This happens because:

$$\lim_{n\to\infty}\mathbb{P}\left(\bigcup_{k=0}^{N} A_k \cap\tau^{-n}(B)\right) = \sum_{k=0}^{N}\lim_{n\to\infty}\mathbb{P}(A_k\cap\tau^{-n}(B)) = \sum_{k=0}^{N}\mathbb{P}(A_k)\mathbb{P}(B) = \mathbb{P}\left(\bigcup_{k=0}^{N} A_k\right)\mathbb{P}(B)$$

And we get $|\mathbb{P}(A\cap\tau^{-n}(B)) - \mathbb{P}(A)\mathbb{P}(B)| < \epsilon$ for sufficiently large n, as we wanted.

Using the $\pi-\lambda$ theorem, we get that $\lim_{n\to\infty}\mathbb{P}(A\cap\tau^{-n}(B)) = \mathbb{P}(A)\mathbb{P}(B)$ for every B cylinder and every borelean A.

Now, let $\mathcal{B}$ be the collection of sets B such that $\lim_{n\to\infty}\mathbb{P}(A\cap\tau^{-n}(B)) = \mathbb{P}(A)\mathbb{P}(B)$ holds for every borelean A. We just proven that $\mathcal{B}$ contain the cylinder sets and now we are going to prove that $\mathcal{B}$ is a $\lambda-$system:

(a) $\Omega \in \mathcal{B}$

$$\lim_{n\to\infty} \mathbb{P}(A\cap\tau^{-n}(\Omega)) = \lim_{n\to\infty} \mathbb{P}(A\cap\Omega) = \lim_{n\to\infty} \mathbb{P}(A) = \mathbb{P}(A) = \mathbb{P}(A)\mathbb{P}(\Omega)$$

(b) $B \in \mathcal{B} \implies B^c \in \mathcal{B}$

$$\lim_{n\to\infty} \mathbb{P}(A\cap\tau^{-n}(B^c)) = \lim_{n\to\infty} \mathbb{P}(A\cap(\tau^{-n}(B))^c) = \lim_{n\to\infty} \mathbb{P}(A\backslash\tau^{-n}(B)) = \lim_{n\to\infty} \mathbb{P}(A) - \mathbb{P}(A\cap\tau^{-n}(B))$$

$$= \mathbb{P}(A) - \lim_{n\to\infty} \mathbb{P}(A\cap\tau^{-n}(B)) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) = \mathbb{P}(A)(1-\mathbb{P}(B)) = \mathbb{P}(A)\mathbb{P}(B^c)$$

(c) If $(B_k)_{k\geq 0}$ is a pairwise disjoint sequence of sets such that $B_k \in \mathcal{B}$ for every k, then $B := \bigcup_{k\geq 0} B_k \in \mathcal{B}$

Defining $B^N := \bigcup_{k>N} B_k$, then:

$$\left|\mathbb{P}(A\cap\tau^{-n}(B)) - \mathbb{P}(A)\mathbb{P}(B)\right| = \left|\mathbb{P}\left(A\cap\left(\tau^{-n}\left(\bigcup_{k=0}^{N} B_k\right)\cup\tau^{-n}(B^N)\right)\right) - \mathbb{P}(A)\mathbb{P}\left(\bigcup_{k=0}^{N} B_k\cup B^N\right)\right|$$

$$= \left|\mathbb{P}\left(\left(A\cap\tau^{-n}\left(\bigcup_{k=0}^{N} B_k\right)\right)\cup\left(A\cap\tau^{-n}(B^N)\right)\right) - \mathbb{P}(A)\left[\mathbb{P}\left(\bigcup_{k=0}^{N} B_k\right)+\mathbb{P}(B^N)\right]\right|$$

$$= \left|\mathbb{P}\left(A\cap\tau^{-n}\left(\bigcup_{k=0}^{N} B_k\right)\right) - \mathbb{P}(A)\,\mathbb{P}\left(\bigcup_{k=0}^{N} B_k\right) + \mathbb{P}\left(A\cap\tau^{-n}(B^N)\right) - \mathbb{P}(A)\mathbb{P}(B^N)\right|$$

$$\leq \left|\mathbb{P}\left(A\cap\tau^{-n}\left(\bigcup_{k=0}^{N} B_k\right)\right) - \mathbb{P}(A)\,\mathbb{P}\left(\bigcup_{k=0}^{N} B_k\right)\right| + \left|\mathbb{P}\left(A\cap\tau^{-n}(B^N)\right) - \mathbb{P}(A)\mathbb{P}(B^N)\right|$$

$$\leq \left|\mathbb{P}\left(A\cap\tau^{-n}\left(\bigcup_{k=0}^{N} B_k\right)\right) - \mathbb{P}(A)\,\mathbb{P}\left(\bigcup_{k=0}^{N} B_k\right)\right| + \mathbb{P}\left(\tau^{-n}(B^N)\right) + \mathbb{P}(A)\mathbb{P}(B^N)$$

Now, given $\epsilon > 0$, we take N such that $\mathbb{P}(\tau^{-n}(B)) = \mathbb{P}(B^N) < \epsilon/3$, and this exists because $\lim_{N\to\infty}\mathbb{P}(B^N) = 0$ by the fact that $\bigcap_{N\geq 0} B^N = \emptyset$ and the semicontinuity. Now, we remark that, for n sufficiently large, we have:

$$\left|\mathbb{P}\left(A\cap\tau^{-n}\left(\bigcup_{k=0}^{N}B_k\right)\right)-\mathbb{P}\left(A\right)\mathbb{P}\left(\bigcup_{k=0}^{N}B_k\right)\right|<\epsilon/3$$

This happens because:

$$\lim_{n\to\infty}\mathbb{P}\left(A\cap\tau^{-n}\left(\bigcup_{k=0}^{N}B_k\right)\right)=\lim_{n\to\infty}\mathbb{P}\left(A\cap\left(\bigcup_{k=0}^{N}\tau^{-n}(B_k)\right)\right)=\sum_{k=0}^{N}\lim_{n\to\infty}\mathbb{P}(A\cap\tau^{-n}(B_k))$$

$$=\sum_{k=0}^{N}\mathbb{P}(A)\mathbb{P}(B_k)=\mathbb{P}(A)\mathbb{P}\left(\bigcup_{k=0}^{N}B_k\right)$$

And we get $|\mathbb{P}(A\cap\tau^{-n}(B))-\mathbb{P}(A)\mathbb{P}(B)|<\epsilon$ for sufficiently large n, as we wanted.

We conclude that $\mathcal{B}\supset\mathcal{A}$, so for every $A,B\in\mathcal{A}$, $\lim_{n\to\infty}\mathbb{P}(A\cap\tau^{-n}(B))=\mathbb{P}(A)\mathbb{P}(B)$. ■

4.7 Phase Transition

Let $\mathfrak{A}=\mathfrak{A}_{\beta,\epsilon}:=\{h;\mu^{+}_{\beta,\epsilon h}(\sigma_0)-\mu^{-}_{\beta,\epsilon h}>0\}$. Our first aim is to use theorem **??** to prove that $\mathbb{P}(\mathfrak{A})>0$. There are two steps from the previous result to the aimed one: we need to (i) rewrite it as a result about the expected value of σ_0 instead of measure of sets and (ii) use the fact that the result holds for every Λ is order to take the thermodynamic limit and get a inequality for the infinite volume measures instead of the finite volume ones. So let's get the hands dirty.

For $h\in\Omega$, we have:

$$\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0)=\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0=1)-\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0=-1)$$

$$\implies\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0)=1-2\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0=-1)$$

Then:

$$\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0)\geq1-2(e^{-c\beta J}+e^{-c/\epsilon^2})\iff\mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0=-1)\leq e^{-c\beta J}+e^{-c/\epsilon^2}$$

We can define:

$$E^{+}_{N}=\{h\in\Omega;\ \mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0=-1)\leq e^{-c\beta J}+e^{-c/\epsilon^2}\}$$

$$=\{h\in\Omega;\ \mu^{+}_{\Lambda_N,\beta,\epsilon h}(\sigma_0)\geq1-2(e^{-c\beta J}+e^{-c/\epsilon^2})\}$$

We define analogously the family of sets (E_N^-). Simple facts from measure theory give us $\mathbb{P}(E_N^+ \cap E_N^-) \geq \mathbb{P}(E_N^+) + \mathbb{P}(E_N^-) - 1$, so:

$$\mathbb{P}(E_N^+ \cap E_N^-) \geq 1 - 2(e^{-c\beta J} + e^{-c/\epsilon^2})$$

Notice that $E_N^+ \cap E_N^-$ correspond to those h such that $\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1) \leq e^{-c\beta J} + e^{-c/\epsilon^2}$ and $\mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = 1) \leq e^{-c\beta J} + e^{-c/\epsilon^2}$ simultaneously, so the sum will be less than $2(e^{-c\beta J} + e^{-c/\epsilon^2})$ for every h in this set. In particular:

$$\mathbb{P}(h;\ \mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1) + \mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = 1) \leq 2(e^{-c\beta J} + e^{-c/\epsilon^2})) \geq 1 - 2(e^{-c\beta J} + e^{-c/\epsilon^2})$$

Using the equality proved above and a similar equality for μ^-, we have:

$$\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0) - \mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0) = (1 - 2\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1)) - (2\mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = 1) - 1)$$

$$= 2 - 2(\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = -1) + \mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0 = 1))$$

As $X \leq k \iff 2 - 2X \geq 2 - 2k$ we are left with:

$$\mathbb{P}(F_N) := \mathbb{P}(h;\ \mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0) - \mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0) \geq 2 - 4(e^{-c\beta J} + e^{-c/\epsilon^2})) \geq 1 - 2(e^{-c\beta J} + e^{-c/\epsilon^2})$$

Consider the intersection $\bigcap_{N\geq 1} F_N$. This set is the set of fields h such that $\mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0) - \mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0) \geq 2 - 4(e^{-c\beta J} + e^{-c/\epsilon^2})$ for every $N \geq 1$. By the FKG inequality, we have that, for $\Lambda \subset \Delta$, $\mu^+_{\Lambda,\beta,\epsilon h}(\sigma_0) \geq \mu^+_{\Lambda,\beta,\epsilon h}(\sigma_0)$ and $\mu^-_{\Lambda,\beta,\epsilon h}(-\sigma_0) \geq \mu^-_{\Lambda,\beta,\epsilon h}(-\sigma_0)$, so $\mu^+_{\Lambda,\beta,\epsilon h}(\sigma_0) - \mu^-_{\Lambda,\beta,\epsilon h}(\sigma_0) \geq \mu^+_{\Delta,\beta,\epsilon h}(\sigma_0) - \mu^-_{\Delta,\beta,\epsilon h}(\sigma_0)$ and hence $(F_N)_N$ is a non-increasing family of sets. By the upper semi-continuity: we have:

$$\mathbb{P}\left(\bigcap_{N\geq 1} F_N\right) = \lim_{N\to\infty} \mathbb{P}(F_N) \geq 1 - 2(e^{-c\beta J} + e^{-c/\epsilon^2})$$

Clearly, if h is in $\bigcap_{N\geq 1} F_N$, then $\lim_{N\to\infty} \mu^+_{\Lambda_N,\beta,\epsilon h}(\sigma_0) - \mu^-_{\Lambda_N,\beta,\epsilon h}(\sigma_0) = \mu^+_{\beta,\epsilon h}(\sigma_0) - \mu^-_{\beta,\epsilon h}(\sigma_0) \geq 2 - 4(e^{-c\beta J} + e^{-c/\epsilon^2})$, and we conclude:

$$\mathbb{P}(h;\ \mu^+_{\beta,\epsilon h}(\sigma_0) - \mu^-_{\beta,\epsilon h}(\sigma_0) \geq 2 - 4(e^{-c\beta J} + e^{-c/\epsilon^2})) \geq 1 - 2(e^{-c\beta J} + e^{-c/\epsilon^2})$$

In particular, if we take c, ϵ and β such that $e^{-c\beta J} + e^{-c/\epsilon^2} < 1/2$, then:

$$\mathbb{P}(h;\ \mu^+_{\beta,\epsilon h}(\sigma_0) - \mu^-_{\beta,\epsilon h}(\sigma_0) > 0) > 0$$

Finally, lets prove that $\mathfrak{A}$ is invariant, that is $\tau^{-1}(\mathfrak{A}) = \mathfrak{A}$.

Let's start by proving that $H^{\eta}_{\Lambda',\epsilon h'}(\sigma') = H^{\eta}_{\Lambda,\epsilon h}(\sigma)$, where $h' = \tau(h)$, $\eta' = \tau(\eta)$ $\sigma' = \tau(\sigma)$ and $\Lambda' = \{i \in \mathbb{Z}^d; i - e_1 \in \Lambda\}$. In fact:

$$H^{\eta'}_{\Lambda',\epsilon h'}(\sigma') = -\sum_{i\sim j\in\Lambda'} \sigma'_i\sigma'_j - \sum_{i\sim j\ i\in\Lambda',\ j\notin\Lambda'} \sigma'_i\eta'_j - \sum_{i\in\Lambda'} h'_i\sigma'_i$$

$$= -\sum_{i\sim j\in\Lambda'} \sigma_{i-e_1}\sigma_{j-e_1} - \sum_{i\sim j\ i\in\Lambda',\ j\notin\Lambda'} \sigma_{i-e_1}\eta_{j-e_1} - \sum_{i\in\Lambda'} h_{i-e_1}\sigma_{i-e_1}$$

$$= -\sum_{i\sim j\in\Lambda} \sigma_i\sigma_j - \sum_{i\sim j\ i\in\Lambda,\ j\notin\Lambda} \sigma_i\eta_j - \sum_{i\in\Lambda} h_i\sigma_i = H^{\eta}_{\Lambda,\epsilon h}(\sigma)$$

Where we used that $i \sim j \iff i - e_1 \sim j - e_1$. With this in mind, we have:

$$Z^{\eta'}_{\Lambda',\beta}(h') = \sum_{\sigma'\in\mathcal{S}_{\Lambda'}} e^{-\beta H^{\eta'}_{\Lambda',h'}(\sigma')} = \sum_{\sigma'\in\mathcal{S}_{\Lambda'}} e^{-\beta H^{\eta}_{\Lambda,h}(\sigma)} = \sum_{\sigma\in\mathcal{S}_{\Lambda}} e^{-\beta H^{\eta}_{\Lambda,h}(\sigma)} = Z^{\eta}_{\Lambda,\beta}(h)$$

And hence:

$$\mu^{\eta'}_{\beta,\epsilon h',\Lambda'}(A') = \frac{1}{Z^{\eta'}_{\Lambda',\beta}(h')} \sum_{\sigma'\in A'} e^{-\beta H^{\eta'}_{\Lambda',h'}(\sigma')} = \frac{1}{Z^{\eta}_{\Lambda,\beta}(h)} \sum_{\sigma'\in A'} e^{-\beta H^{\eta}_{\Lambda,h}(\tau^{-1}(\sigma'))}$$

$$= \frac{1}{Z^{\eta}_{\Lambda,\beta}(h)} \sum_{\sigma\in A} e^{-\beta H^{\eta}_{\Lambda,h}(\sigma)} = \mu^{\eta}_{\beta,\epsilon h,\Lambda}(A)$$

Where $A = \tau^{-1}(A)$. Now, since $\tau^{-1}(\{\sigma; \sigma_1 = a\}) = \{\sigma; \sigma_0 = a\}$ for $a \in \{-1, 1\}$, we have:

$$\mu^{\eta}_{\beta,\epsilon h,\Lambda}(\sigma_0) = \mu^{\eta}_{\beta,\epsilon h,\Lambda}(\sigma_0 = 1) - \mu^{\eta}_{\beta,\epsilon h,\Lambda}(\sigma_0 = -1) =$$

$$\mu^{\eta'}_{\beta,\epsilon h',\Lambda'}(\sigma_1 = 1) - \mu^{\eta'}_{\beta,\epsilon h',\Lambda'}(\sigma_1 = -1) = \mu^{\eta'}_{\beta,\epsilon h',\Lambda'}(\sigma_1)$$

As we will be interested in $\eta = \pm 1$, $\eta' = \eta$, so we have:

$$\mu^{\eta}_{\beta,\epsilon h}(\sigma_0) = \lim_{N\to\infty} \mu^{\eta}_{\beta,\epsilon h,\Lambda_N}(\sigma_0) = \lim_{N\to\infty} \mu^{\eta}_{\beta,\epsilon h',\Lambda'_N}(\sigma_1) = \mu^{\eta}_{\beta,\epsilon h'}(\sigma_1)$$

Because $\mu^{\eta}_{\beta,\epsilon h}$ does not depend on the sequence Λ, we may take either $(\Lambda_N)_N$ or $(\Lambda'_N)_N$. Furthermore, these states are translation invariant, which implies that $\mu^{\eta}_{\beta,\epsilon h'}(\sigma_1) = \mu^{\eta}_{\beta,\epsilon h'}(\sigma_0)$, and we conclude, finally:

$$\mu^{+}_{\beta,\epsilon h}(\sigma_0) - \mu^{-}_{\beta,\epsilon h}(\sigma_0) = \mu^{+}_{\beta,\epsilon h'}(\sigma_0) - \mu^{-}_{\beta,\epsilon h'}(\sigma_0)$$

Which means that $h \in \mathcal{A} \iff h' \in \mathcal{A}$, so $\tau^{-1}(\mathcal{A}) = \mathcal{A}$.

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