KMS states and C* Algebras of Groupoids

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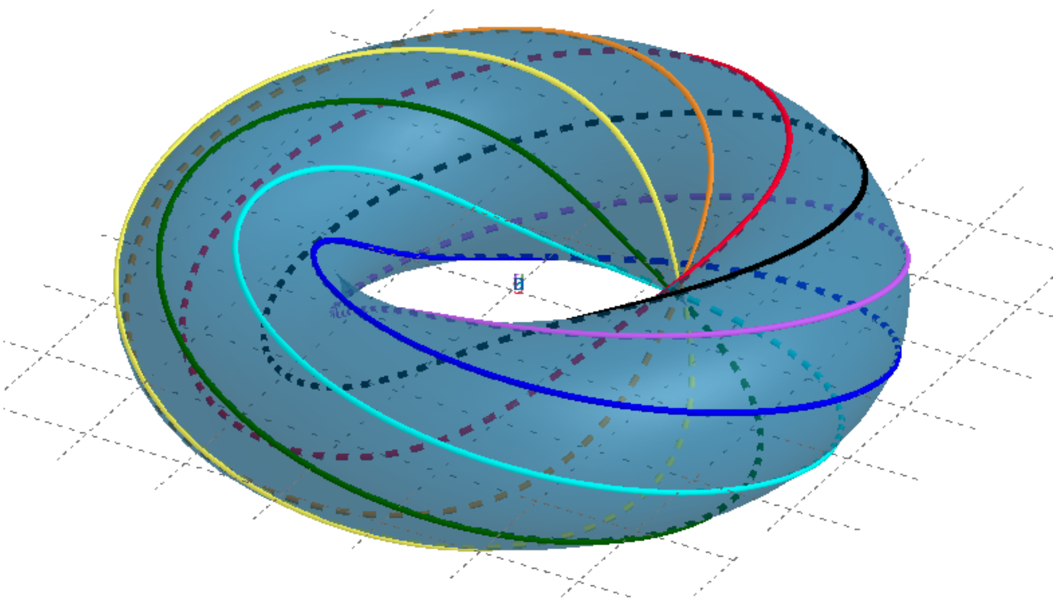

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Contents

Preface

The goal of this text is to present some of the main results of Renault and Kumjian [KR06] and, consequently, develop the preliminary theory. This includes some notions of groupoids, C* algebras and their dynamical systems and a basic of KMS states.

The script is the following: we associate a groupoid to a given dynamical system and construct a C*-algebra over this groupoid. Then, we endow the algebra with a group of automorphisms and investigate the existence and uniqueness of its equilibrium (KMS) states. The techniques and tools used in the proofs will be borrowed from thermodynamic formalism, so part of the work will be to relate KMS states with eigenmeasures of the Ruelle's operator. In our view, this programme is very promising, once we can import results of one theory to another.

The definition of the main objects, like groupoids and C*-algebras and their most basic properties are supposed to be known. Although we describe *en passant* the construction of the C* algebra of a groupoid, all those details may be found in many sources, and we particularly recommend [Sim17], [Fra18], [Lim19] and [Ras20].

As already said, the main results concern the existence and uniqueness of KMS states. This type of result have been investigate for a long time, since such states have a deep physical meaning in quantum statistical mechanics, for example, the non-uniqueness of a KMS state is associated to a phase transition. To cite some concrete situation, we affirm that the dynamics of a quantum lattice system may be given by what is called an approximately inner group of automorphism (e.g. [Rob67]) and some result about their KMS states may be found in [PS75].

Chapter 1

Rudiments of Thermodyamic Formalism

Since the seminal work [Rue68] of Ruelle, the machinery envolving the transfer operator and Ruelle's theorem have been widely developed. Although the operator was already known before him, his usage and results have inspired many prolific mathematicians. As his original motiviation was an one-dimensional lattice gas, the so-called Ruelle-Perron-Frobenius theorem was originally very well suited in the shift space framework (we recommend [Bow75] for an overview).

It did not take so long for the theory to be generalized. One of the first succesful generalizations may be found in [Wal78], but we refer to [Wal01] for the full version we are going to use. Before we properly enunciate the results, we must present the definition and basic properties of some types of dynamical systems:

Definition 1.1. A dynamical system (X, T) is said expansive if there is ϵ (expansive constant) such that

$$\rho(T^n(x), T^n(y)) > \epsilon$$

for some $n \geq 0$, where ρ is the metric. It is said exact if, for every nonempty open set U, there is $n > 0$ such that $T^n(U) = X$. It satisfies the weak specification condition if, for every, $\epsilon > 0$ there is $n > 0$ such that $T^{-n}(x)$ is $\epsilon-$dense in X for every $x \in X$.

Notice that the last definition intend to translate a certain notion of super non-injectivity. We have the following result:

Proposition 1.1. *If $T : X \to X$ is a local homeomorphism, then T is exact if, and only if T satisfies the weak specification condition.*

The proof may be founded in [KR06]. The generalized setting will be precisely a compact metric space X with an expansive, exact, local homeomorphism T.

Example 1. *The shift space in n letters (Σ_n, σ) is expansive and exact. In fact, for every pair of distinct points x, y, we can iterate σ until $x_0 \neq y_0$, so eventually x, y will be have a* *distance of 1 between them. Moreover, the open balls are the cylinders, and if we iterate the image of any cylinder by σ a certain amount, we will obtain the whole space again. For the sake of simplicity, every concrete example of a full shift from now on will be in 2 letters, but the adaptations for other number of letters are straightforward.*

Example 2. *One can easily verify that the system composed by the unit complex circle $\mathbb{T}$ and the local homeomorphism $x \mapsto x^2$ is expansive and exact. Actually, this holds true for $(\mathbb{T}, F_p)$, where $x \mapsto x^p$, for every $p \geq 2$.*

Definition 1.2. A function $\varphi \in \mathcal{C}(X, \mathbb{R})$ satisfies the Bowen condition if there are $\delta, C > 0$ such that:

$$d(T^i x, T^i y) < \delta, 1 \leq i \leq n-1 \implies \sum_{i=0}^{n-1} \varphi(T^i x) - \varphi(T^i y) \leq C$$

$\forall x, y \in X, n > 0$

We may define the transfer operator in the most obvious way. In the original setting, the summation was naturally finite, and this feature is guaranteed by the hypothesis on (X, T). We are, then, ready to state the theorem proved in [Wal01].

Theorem 1.2. *Let $T : X \to X$ be an exact, expansive, local homeomorphism and $\varphi \in \mathcal{C}(X, \mathbb{R})$. Then there is a probability measure ν and $\lambda > 0$ such that $\mathcal{L}_\varphi^* \nu = \lambda \nu$ and:*

$$\frac{1}{n} \log\big(\mathcal{L}_\varphi^n 1\big)(x) \to P(T, \varphi) = \log \lambda$$

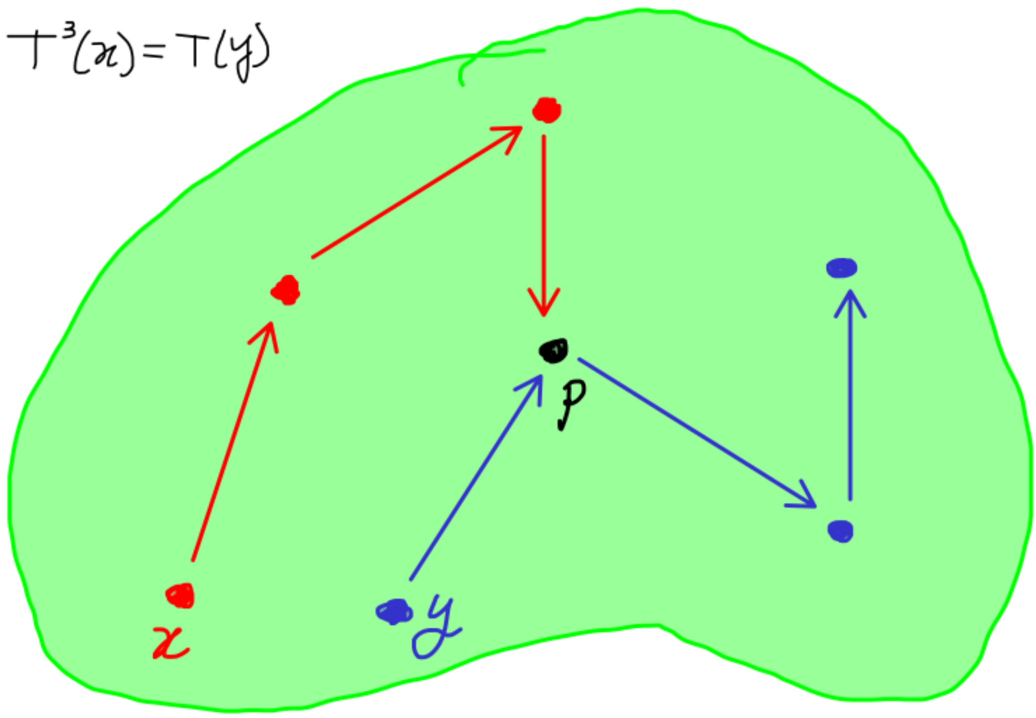
uniformly. If ϕ satisfies Bowen condition, such ν is unique.

Chapter 2

Groupoids

2.1 Renault-Deaconu

Let (X,T) be a dynamical system. Consider the following situation:



Notice that x e y almost met in p, if there wasn't a delay of two units of time. We can represent this event by a triple $(x,2,y)$, and then construct the set of such triples, which we are going to call

$$G(X,T) := \{(x,m-n,y), T^m x = T^n y\}$$

Notice also that, if x,y have a delay of k units of time and y,z have a delay of l units, then

x and z have a delay of $k+l$ units, so (x,k,y) and (y,l,z) give origin to $(x,k+l,z)$. In a similar manner, (x,k,y) implies the existence of $(y,-k,x)$. We can think of these properties of closure of $G(X,T)$ as operations: the first one being a kind of multiplication and the second one a kind of inverse. With these observations, one could think of $G(X,T)$ as a group. However, G is not formally a group, once we cannot apply the multiplication to any pair of elements. $(x,k,y)\cdot(w,l,z)$ only makes sense if $y=w$. A set which has operations similar to those of a group, with exception that the multiplication is not defined in all $G\times G$ is called a groupoid.

The notion of a groupoid was first introduced by Brandt in the 1920s, and its definition was developed in the following years, with a few modifications. We are not going to provide a detailed introduction to these objects, but we refer to [Ren80]. We remark that there is a category theoretical approach to groupoids. In this setting, a groupoid is simply a small category such that every morphism is an isomorphism. Although this approach may look rather technical and even unnecessary, it provides us an interesting view on groupoids, in which each element is an arrow (morphism), each element of the unit space may be identified with the objects by means of the identity morphism and the functions s and r turn out to be just the domain and codomain functions respectively, so that the most basic properties become very intuitive. We recommend [Ras20] for a more detailed explanation of the view with arrows.

Example 3. *Let $R\subset X\times X$ be an equivalence relation on X. R may be seen as a groupoid with $R^2=\{((x,y),(z,w))\in R\times R; y=z\}$, $(x,y)\cdot(y,z)=(x,z)$ and $(x,y)^{-1}=(y,x)$.*

Example 4. *Let G be a group acting on a space X. Two elements $(x,g),(y,h)\in X\times G$ will be composable if $y=gx$. The composition will be defined by: $(x,g)(y,h)=(x,gh)$. The inversion will be $(x,g)^{-1}=(gx,g^{-1})$. This groupoid is called "transformation group". Notice that $r(x,g)=(x,e)$ and $s(x,g)=(gx,e)$.*

Particularly, $G(X,T)$ was introduced in 1995 by Valentin Deaconu [Dea95], following ideas from Renault [Ren80]. He first considered the case where X was a compact Hausdorff space and T continuous and surjective.

For $k\geq 0$, we define $R_k=\{(x,0,y)\in G; T^kx=T^ky\}$ and:

$$R_\infty=\bigcup_{k\geq 0}R_k$$

Notice that $R_0=G^0$ and every R_k is a groupoid (arising from an equivalence relation).

Also, notice that, if $T^n(x)=T^m(y)$, then $T^{k+n}(x)=T^{k+m}(y)$, but the reciprocal is not necessarily true. The non-injectivity allows one to have $T^n(x)\neq T^m(y)$ but $T^{k+n}(x)=T^{k+m}(y)$. This observation implies that $\{R_k\}_k$ is an increasing sequence of sets. In the case of a bijection, we would clearly have a constant sequence $R_k=G^0$, for every k.

Example 5. *Consider the system $(\mathbb{T},F_2)$. Given $x=e^{i\theta}\in\mathbb{T}$, it is easy to find the set of y such that $F_2(x)=F_2(y)$, in other words, the roots for the equation $e^{i2\theta}-z^2$. Since it is a second order equation, there are exactly two solutions, being $e^{i\theta}$ and $-e^{i\theta}=e^{i(\theta+\pi)}$. Thus, $R_1=\{(x,0,y); y=\pm x\}=\{(x,0,x); x\in\mathbb{T}\}\cup\{(x,0,-x); x\in\mathbb{T}\}$, and clearly this set can be regarded as the union of two circles. Notice how it contains G^0 (the first of the two circles). By similar reasonings, we may get to the fact that $R_2=\{(x,0,y); y=xe^{ik\pi/2}, k=0,1,2,3\}$ (four circles) and, more generally:*

$$R_n = \left\{(x,0,y); y = xe^{\frac{2\pi ki}{2^n}} \text{ with } k = 0, ..., 2^n - 1\right\}$$

And one can view it as 2^n circles. This image of various circles will be resumed and more explored once we introduce topology on groupoids.

Example 6. *For the full shift in two symbols, (Σ_2, σ), clearly*

$$R_n = \{(w_1x, 0, w_2x); w_1, w_2 \text{ are words of length } n\}$$

and it can be thought, again, as 2^n copies of the shift.

More generally if Σ_A is an arbitrary Markov shift with transition matrix A, it is easily to see that $G(\Sigma_A, \sigma) = \{(ax, |a| - |b|, bx), \text{ where } ax \text{ and } bx \text{ are admissible}\}$.

Proposition 2.1. *If T is a bijection, then $G(X,T)$ is isomorphic to the transformation group $X \times_T \mathbb{Z}$, with the action being $(k,x) = T^k(x)$*

Proof. Consider the mapping $\Pi : X \times_T \mathbb{Z} \to G(X,T)$ given by $\Pi(x,k) = (x, -k, T^k(x))$.

This map is a homomorphism. Indeed:

$$\Pi(x,k)\Pi(T^k(x), l) = (x, -k, T^k(x))(T^k(x), -l, T^{k+l}(x))$$
$$= (x, -k-l, T^{k+l}(x)) = \Pi(x, k+l) = \Pi((x,k)(T^k(x), l))$$

And:

$$\Pi((x,k)^{-1}) = \Pi(T^k(x), -k) = (T^k(x), k, x) = (x, -k, T^k(x))^{-1} = (\Pi(x,k))^{-1}$$

□

This identification will simplify some reasoning in the future.

2.2 Basic Notions and Properties

Definition 2.1. We denote $G_x = s^{-1}(x)$, $G^x = r^{-1}(x)$, and call them respectively, s-fiber and r-fiber. Also $G^x_y = G^x \cap G_y$. G^x_x is called isotropy group for x and

$$\text{Iso}(G) = \bigcup_{x \in G^0} G^x_x = \{g \in G; r(g) = s(g)\}$$

is called the isotropy bundle.

We remark that G^x_x it is always a group, which justifies both the name *isotropy group* and *isotropy bundle*. Notice that G^x_x is the set of arrows going in and out the same point x so, in some sense, they fix the point x (recall the definition of isotropy group in group theory). The name is even more justified considering the transformation groups. On the other hand, each isotropy group may be seen as a fiber over G^0, so the union may be seen as a bundle of groups. It is just the set of arrows that come in and out the same point, whatever the point is.

Proposition 2.2. *Let (X, T) be a dynamical system and $G(X, T)$ the Renault-Deaconu groupoid associated. Given $x \in G^0$, the isotropy group G_x^x is a singleton or infinite.*

Proof. Suppose G_x^x is not a singleton. Then there is $n \neq 0$ such that $(x, n, x) \in G_x^x$. In other words, there exists m such that $T^{m+n}(x) = T^m(x)$. But this implies $T^{m+2n}(x) = T^n(T^{n+m}(x)) = T^n(T^m(x)) = T^{n+m}(x) = T^m(x)$, hence $(x, 2n, x) \in G_x^x$. By a simple induction, we see that $(x, kn, x) \in G_x^x$, for all k integer.

A much more direct way is recall that G_x^x is a group, and hence the composition composição $(x, n, x)(x, n, x) = (x, 2n, x)$ is in G_x^x, then proceed inductively. □

As it is clear, G_x^x is not necessary unitary, but we have interesting results when it occurs, we are going to briefly investigate them in what follows.

Definition 2.2. A groupoid is said to be principal if the map $G \ni g \mapsto (s(g), r(g)) \in G^0 \times G^0$ is injective.

Proposition 2.3. *A groupoid G is principal if, and only if, Iso$(G) = G^0$*

Proof. Recall that $g \in G^0$ iff $s(g) = r(g) = g$, so $G^0 \subset \text{Iso}(G)$. Now, suppose that $g \in \text{Iso}(G)$, and let $x = r(g) = s(g) \in G^0$. This implies that $r(x) = s(x) = x$ and by the injectivity, $g = x$, so $g \in G^0$. Reciprocally, let $(r(g), s(g)) = (r(h), s(h))$. $s(gh^{-1}) = s(h^{-1}) = r(h) = r(g) = r(gh^{-1})$, so $gh^{-1} \in \text{Iso}(G)$. By hypothesis, this means that $gh^{-1} \in G^0$. Thus, $gh^{-1} = r(h) = hh^{-1} \implies g = h$ by cancellation. □

Proposition 2.4. *A groupoid G is algebrically isomorphic to an equivalence relation if, and only if, G is principal*

The distinction between principal groupoid and equivalence relations will be of topological order.

Example 7. *The subgroupoids R_n are principal, since they arise from equivalence relations, as already stated.*

Proposition 2.5. *Let (X, T) be a dynamical system and $G(X, T)$ the Renault-Deaconu groupoid associated. Then, G is principal if, and only if, X has no periodic points.*

Proof. Suppose that G is not principal. So there is $(x, k, x) \in G$ with $k \neq 0$. But this implies the existence of n such that $T^{n+k}(x) = T^n(x)$, thus $T^n(x)$ is a point of period k. Reciprocally, suppose there is a point $k-$ periodic x. Then, it holds that $T^k(x) = x$, that is, $(x, k, x) \in G$, and therefore G is not principal. □

As with groups, we may construct a cohomology theory with groupoids, with little adaptations. We are not going to develop such a theory here, but we remark that this proceeding provide helpful tools such as n-cocycles, being 1-cocyles and 2-cocyles the most important ones. In our case, a 1-cocycle will be simply a kind of homomorphism:

Definition 2.3. Let G be a groupoid. A 1-cocycle is a homomorphism from G to the additive group of $\mathbb{R}$.

As usual homomorphisms, it is straightforward that $c(x) = 0$ for every $x \in G^0$ and $c(g^{-1}) = -c(g)$, for every $g \in G$.

2.3 Topological Groupoids

Our purpose is to construct a C* algebra of a groupoid, and an intermediary step for it will be the algebra of the continuous functions with compact support, so a topology on G will be essential. The requirements in order to assure the compatibility of the algebraic and topologic structures will be the most natural ones:

Definition 2.4. Let G be both a groupoid and a topological space. G will be a topological groupid provided $^{-1} : G \to G$ is continuous and $\cdot : G^2 \to G$ is continuous in the subspace topology of the product topology of $G \times G$

We are going to give some examples of topological groupoids without proof.

Example 8. *If $X \times G$ is a transformation group, then it may be regarded as a topological group in the product topology provided X and G are topological spaces such that X is a second-coutable locally compact Hausdorff and G is locally compact.*

Example 9. *For a second-countable Hausdorff space X with an equivalence relation R, one can prove that R is a topological groupoid with the topology being induced from $X \times X$.*

Example 10. *From the above example, if $G(X,T)$ is a Renault-Deaconu groupoid, then $\{R_n\}$ is a sequence of topological subgroupoids, and we can make R_∞ a topological groupoid as well, with the inductive limit topology. Take a look at the appendix A for a discussion about this kind of topology.*

Proposition 2.6. *G^0 is closed if, and only if, G is Hausdorff.*

Proof. Let (x_i) be a net in G^0 converging to $x \in G$. As r is continuous, $x_i = r(x_i) \to r(x)$. If G is Hausdorff, the limit is unique, so $x_i \to r(x) \in G^0$. Reciprocally, let (x_i) be a net converging to a and b. By continuity, $x_i^{-1}x_i \to a^{-1}b$. But as $x^{-1}x_i = s(x_i) \in G^0$, if G^0 is closed, then $a^{-1}b \in G^0$, so $a = b$ and the limit is unique. This already implies that G is Hausdorff.

□

As the last proposition suggests, it will be of extreme importance for our purposes to require more hypothesis, like local compactness, Hausdorff and some authors even include them in the definition of topological groupoid. It is clear that r and s are continuous functions in every topological groupoid, but another essential requirement is to ask them to be local homeomorphisms.

Definition 2.5. A topological groupoid is said to be étale if r and s are local homeomorphisms. $\mathcal{U} \subset G$ is an open bisection if r and s are simultaneously homeomorphims when restricted to it.

That is so important that it is even easier to drop the Hausdorff condition than the etalicity. It is not hard to see that is enough to ask r to be a local homeomorphism.

Proposition 2.7. *If G is étale, then the collection of open bisections is a basis for the topology, G^0 is open and G^x, G_x are discrete for every x.*

Proof. The first one is straightforward. For the second, consider $x \in G^0$. If G is étale, there is an open neighborhood $\mathcal{U}$ of x where r is a homeomorphism. Notice that $r(\mathcal{U}) \subset G^0$ is open and contains x, so x is an interior point. Finally, given $g \in G^x$, there is $\mathcal{U}$ with $g \in \mathcal{U}$. If $h \in G^x \cap \mathcal{U}$, then $r(h) = x$, but then $h = g$ by the injectivity of r in $\mathcal{U}$. The proof of G_x is completely analogous. □

Notice that, since r and s are injective in G^0, one may show that G^0 is an open bisection if G is étale.

Proposition 2.8. *If $\mathcal{U}, \mathcal{V}$ are open bisections, then so are $\mathcal{U}\mathcal{V} = \{gh : g \in \mathcal{U}, h \in \mathcal{V}, (g,h) \in G^{(2)}\}$ and $\mathcal{U}^{-1} = \{g^{-1} : g \in \mathcal{U}\}$.*

With respect to the Reanault-Deaconu groupoid, we are going to consider the topology whose basis is formed by set of the form: $W(n,m,U,V) = \{(x, n-m, y); x \in U, y \in V, T^n(x) = T^m(y)\}$, where U and V are open. One can show that:

$$W(n_1, m_1, U_1, V_1) \cap W(n_2, m_2, U_2, V_2) = W(\min\{n_1, n_2\}, \min\{m_1, m_2\}, U_1 \cap U_2, V_1 \cap V_2)$$

So that:

Proposition 2.9. *The set of $W(n,m,U,V)$ is a basis for some topology.*

Another remarkable and easy fact is the following: if $U' \subset U$ and $V' \subset V$, then clearly $W(n,m,U',V') \subset W(n,m,U,V)$.

One could reasonably asks himself about the relation of this topology to the (subspace topology of the) product topology. To address this question, we remark the following simple relation:

$$W(n,m,U,V) \subset W(k+n, k+m, U, V), \quad k > 0$$

If we are dealing with a bijection, the equality clearly holds. In this case, only one integer matters and, moreover: $W(k,U,V) = \{(x,k,T^k(x)); x \in U, T^k(x) \in V\} = \{(x,k,T^k(x)); x \in U \cap T^{-k}(V)\}$. So, if T is continuous, only one open set matters. The basis can be, thus, written as $W(k,U) = \{(x,k,T^k(x)); x \in U\}$. This consideration leads to the fact that, if T is a homeomorphism, then $G(X,T)$ and $X \times_T \mathbb{Z}$ are not only algebrically isomorphic, but are also homeomorphic.

This considerations must give the reader a clue about the relation between the defined topology and the product one: the topology we have actually put on $G(X,T)$ explicitly depends on n, m, whereas only the difference matters for the product one. Consider, for example, the set $X = \{1, 2, 3\}$ with the dynamics $T(1) = T(2) = 1$ and $T(3) = 2$ and topology $\tau = \{\emptyset, \{1\}, X\}$. Then it is a matter of fact that $W(1, 0, X, \{1\}) = \{(1, 1, 1), (2, 1, 1)\}$, while only $\{(1, 1, 1), (2, 1, 1), (3, 1, 1)\} = X \times \{1\} \times \{1\} = W(2, 1, X, \{1\})$ is open in the product topology, so it is coarser.

We don't need to consider all the open sets U, V, it suffices to consider the elements of some basis for the topology:

Proposition 2.10. *Let $\mathcal{B}$ be a basis for the topology of X. Then, the collection of sets $W(n, m, B_1, B_2)$ with $B_1, B_2 \in \mathcal{B}$ is a basis for the topology of $G(X, T)$.*

Proof. Let $A \subset G(X, T)$ be an open set. It is enough to show that, for every $p \in A$, there is $W(n, m, B_1, B_2)$ such that $p \in W(n, m, B_1, B_2) \subset A$. Well, we must have n, m, U, V such that $p \in W(n, m, U, V) \subset A$. If $p = (x, n - m, y)$, then there must exist B_1, B_2 such that $x \in B_1 \subset U$ and $y \in B_2 \subset V$. I claim that $W(n, m, B_1, B_2)$ satisfies the desired requirements. We clearly have $p \in W(n, m, B_1, B_2)$ because $T^n(x) = T^n(y)$ by hypothesis. Now, if $q = (x', n - m, y') \in W(n, m, B_1, B_2)$, then $T^n(x') = T^m(y')$ and $x' \in B_1 \subset U$ and $y' \in B_1 \subset V$, hence $q \in W(n, m, U, V)$ and we are done. $\square$

Example 11. *For $G(\Sigma_A, \sigma)$, let a, b be two admissible words and n, m such that $n \geq |a|$ and $m \geq |b|$. Then, $W(n, m, [a], [b]) = \{(aw_1x, n - m, bw_2x) \in G(\Sigma_A, \sigma); |w_1| = n - |a|, |w_2| = m - |b|\}$ is a class of open sets. We define $Z(a, b) := W(|a|, |b|, [a], [b])$. It is not too hard to convince yourself that the collection of $Z(a, b)$ forms a basis for the topology. For example, in the case $n \geq |a|$ and $m \geq |b|$, for every p in an open set U, we have $p \in W(n, m, [a], [b])$, so we may write $p = (aw_1x, n - m, bw_2x) \ni W(|aw_1|, |bw_2|, [aw_1], [bw_2]) = W(n, m, [aw_1], [bw_2]) \subset W(n, m, [a], [b])$. The remaining cases is not so different. We also define $Y_k := \{(kx, 1, x) \in G(\Sigma_A, \sigma)\}$.*

We are not going to prove the following proposition, but we refer to [Ras20], §4.5

Proposition 2.11. *If X is Hausdorff, $G(X, T)$ is étale and Hausdorff. If X is locally compact, then $G(X, T)$ is also locally compact. If X is second countable, $G(X, T)$ is also second countable.*

Proposition 2.12. *G^0 and X have the same topology.*

Corollary 2.12.1. *G^0 is compact*

Proof. As X is compact, G^0 is compact in its topology. But as G^0 is open, it is also compact in the topology of $G(X, T)$ $\square$

2.4 Haar Systems

In the next section, we are going to deal with the task of constructing a C*-algebra of a groupoid. In some sense, this construction tries to resemble to some extent the C*-algebra of a group. The algebra of a group is usually considered to be $L^1(G)$. For groups (specifically, for locally compact ones), this task is facilitated due to Haar theorem. This theorem states the existence and uniqueness of a regular measure in G that is translational invariant: $\mu(gG) = \mu(G), \forall g \in G$.

When dealing with groupoids, the situation is considerably wilder. The best we can do is to consider a family of measures $\{\lambda^x\}_x$ indexed by the elements of G^0, with each one being supported in G^x, and called Haar system. We require some continuity hypothesis:

$$x \mapsto \int_G f(g) d\lambda^x(g)$$

must be continuous for every $f \in \mathcal{C}_c(G)$ and the connection with Haar measures is made by the following condiiton:

$$\int_G f(hg) d\lambda^{s(h)}(g) = \int_G f(g) d\lambda^{r(h)}(g)$$

And it becomes clearer when we write the translational invariance in the integral form. A Haar measure for group satisfies:

$$\int_G f(hg) d\mu(g) = \int_G f(g) d\mu(g)$$

Chapter 3

C* algebra

3.1 Full algebra of a groupoid

We are going to consider in this section a LCH (locally compact Hausdorff) second countable étale topological groupoid, although some of these assumptions are not strictly necessary (see [Ren80]). We are not going to provide some technical details, but whenever we do so, they may be easily found in [Ras20], for example.

As already mentioned, the construction resembles the case of a (LCH) group. For the sake of motivation, we are going to briefly describe the construction of the C* algebra of a group. We will start with the case of a finite discrete group $G = \{g_1, ..., g_n\}$. In this case, we consider the free vector space generated by G, that is the set of formal linear combinations of elements of G and which is isomorphic to the set of complex-functions on G. To turn this space into an algebra, with an bilinear product which extends the group operation, we have to define:

The first step is to consider the space of continuous functions of compact support $\mathcal{C}_c(G)$ and endow it with operations in order to be a *-algebra:

$$(f_1 \cdot f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2)$$

$$f_1^*(g) = \overline{f_1(g^{-1})}$$

Maybe the first thing one notice about the convolution is the cumbersomeness of the set over which the summation is taken. We will seek ways to minimize this inconvenient. For some functions, this summation will collapse to only one term. But generally, we can rewrite in the more practical form, that is basically a change of variables:

$$(f_1 \cdot f_2)(g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2) = \sum_{h \in G^{r(g)}} f_1(h) f_2(h^{-1} g) = \sum_{h \in G_{s(g)}} f_1(g h^{-1}) f_2(h)$$

In this form, it is more clear that the summation is finite: As $G^{r(g)}$ is discrete, and the support of f_1 is compact, the non-vanishing terms will be in $G^{r(g)} \cap \text{supp} f_1$, finite.

We are not going to show in details that these operations are well-defined (the result has compact support) and satisfies the requirements of a *-algebra, but the procedure is usually the following: we first show that these properties hold for functions with support in open bisections and then generalize noticing that we can write every $f \in \mathcal{C}_c(G)$ as a finite sum of functions with support in open bisections. This, in turn, uses the "partition of unity" property and the fact that the open bisections forms a basis. This provides us a first glance of the importance of etalicity. There are intermediary steps that we will enunciate for later use.

We remark that $\mathcal{C}_c(U) \subset \mathcal{C}_c(G)$ if U is open. Indeed, if f has compact support in U, then the support is also compact in G if U is open. Frequently, U will be an open bisection or the unit space.

Proposition 3.1. *If $f_1 \in \mathcal{C}_c(\mathcal{U}_1)$ and $f_2 \in \mathcal{C}_c(\mathcal{U}_2)$, then $f_1 \cdot f_2 \in \mathcal{C}(\mathcal{U}_1\mathcal{U}_2)$ and $f_1^* \in \mathcal{C}(\mathcal{U}^{-1})$.*

Proposition 3.2. *The operations of convolution and involution restricted to $\mathcal{C}_c(G^0)$ are the usual pointwise multiplication and complex conjugation. Moreover, $\mathcal{C}_c(G^0)$ is an union of C*-algebras.*

The following example is related to the Heisenberg picture of quantum mechanics.

Example 12. *Let n be a positive integer and $G_n = \{(i,j); i,j = 1,...,n\}$ with $G^2 = \{((i,j),(k,l)) \in G_n \times G_n; j = k\}$, $(i,j)\cdot(j,k) = (i,k)$ and $(i,j)^{-1} = (j,i)$. If we endow G_n with the discrete topology, G will be a compact Hausdorff étale topological groupoid. Trivially, $C_c(G)$ is identified with $M_n(\mathbb{C})$, with $f \mapsto F$ such that $f(i,j) = F_{ij}$. It is not hard to see that the operations also coincide.*

Once we have a *-algebra, the last step is to obtain a C*-norm. We will endow $\mathcal{C}_c(G)$ with the following norm:

$$||f|| = \sup \Pi_f, \qquad \Pi_f = \{x \in \mathbb{R}; x = ||\pi(f)||, \pi : \mathcal{C}_c(G) \to \mathcal{B}(H) \text{ *-repres.}\}$$

Π_f is clearly non-empty for every f, but we still need to show that it is bounded. Before that, we emphasize that all the axioms of C*-seminorm are very straightforward to be verified, and they all follow from the fact that π are representations and the elementary properties of the supremum. To show that $||f|| > 0$ if $f \neq 0$, however, more work must be done. The C*-algebra of a groupoid is, then, the completion of $\mathcal{C}_c(G)$ with respect to this norm, and denoted by $C^*(G)$.

Proposition 3.3. Π_f *is bounded for every $f \in \mathcal{C}_c(G)$*

Proof. As usual, we are going to consider subclasses first. Take $f \in \mathcal{C}_c(G^0)$. Then f is in some C*-algebra $\mathcal{C}(K)$. As we know that a homomorphism between C*-algebras decreases the norm, we have that $||\tilde{\pi}(f)|| \leq ||f||_\infty$ (with $\tilde{\pi}$ being the restriction of π to $\mathcal{C}_c(G^0)$), for every π, so Π_f is bounded.

If $f \in \mathcal{C}_c(\mathcal{U})$, then $f^* \cdot f \in \mathcal{C}_c(G^0)$. In fact, by proposition 3.1, $f^* \cdot f \in \mathcal{C}_c(\mathcal{U}^{-1}\mathcal{U})$, and

$$\mathcal{U}^{-1}\mathcal{U} = \{gh; g \in \mathcal{U}^{-1}, h \in \mathcal{U}, s(g) = r(h)\} = \{g^{-1}h; g,h \in \mathcal{U}, r(g) = r(h)\} = \{g^{-1}g; g \in \mathcal{U}\} = s(\mathcal{U}) \subset G^0$$

, by the injectivity of r in $\mathcal{U}$. So, as $\mathcal{B}(H)$ is a C*-algebra:

$$||\pi(f)||^2 = ||\pi(f)^*\pi(f)|| = ||\pi(f^* \cdot f)|| \leq ||f^* \cdot f||^\infty$$

Finally, using again the trick of the partition of unity:

$$||\pi(f)|| = ||\pi(\sum f_i)|| = ||\sum \pi(f_i)|| \leq \sum ||\pi(f_i)|| \leq \sum \sqrt{||f_i^* \cdot f_i||_\infty}$$

□

We note that $||f^* \cdot f||_\infty = ||f||_\infty^2$, for every $f \in \mathcal{C}_c(\mathcal{U})$ [Fra18].

Proposition 3.4. $|| \cdot ||$ *is a C*-norm*

Again, we are not going to make explicit the details, but the ideia is to construct a concrete representation where $||\pi(f)|| > 0$. If $f > 0$, then $f(y) \neq 0$ for some y. Let $x = s(y)$, then, the Hilbert space considered will be:

$$H = \ell^2(G_x) = \left\{ \{z_g\}_{g \in G_x} \in \mathbb{C}^{G_x}; \sum_{g \in G_x} |z_g|^2 < +\infty \right\}$$

$$\langle z, w \rangle = \sum_{g \in G_x} z_g \overline{w_g}$$

And the represantation:

$$(\pi^x(f)(z))_g = \sum_{h_1 h_2 = g} f(h_1) z_{h_2}$$

So if we take $z = \delta_x$, the $y-$th component of $\pi^x(f)(\delta_x)$ would be nonzero, so $\pi^x(f) \neq 0$ and then $||\pi^x(f)|| > 0$. Indeed, by a simple change of variables:

$$(\pi^x(f)(z))_y = \sum_{h \in G^{r(y)}} f(h) z_{h^{-1}y}$$

By definition of z the non-vanishings terms will be those with $h^{-1}y = x$. But then, $\implies h^{-1}y = y^{-1}y \implies h^{-1} = y^{-1}$, so we will only have one term:

$$(\pi^x(f)(z))_y = f(y) z_x = f(y) \neq 0$$

As we wanted. Rigorously, we should prove that π^x is a *-representation and $\pi^x(f)$ is bounded.

For some $x \in G^0$ arbitrary, the representation π^x is important and is used to construct another type of C* algebra of groupoids. Indeed, the algebra constructed here is known as full C* algebra of G. However, one can consider the direct sum of $\ell^2(G_x)$ (actually, the completion),

$\ell^2(G)$ and a representation π_λ of $\mathcal{C}_c(G)$ to $\mathcal{B}(\ell^2(G))$ (called the regular representation), define $||f||_r = ||\pi_\lambda(f)||$ and the completion of $\mathcal{C}_c(G)$ with respect to this norm will be called the reduced C* algebra of G, denoted by C_r^*. This norm has the following property:

$$||f||_r = \sup_{x \in G^0} ||\pi^x(f)||$$

The reduced algebra may always be seen as the quotient of the full by some ideal. In some cases (which has to do with amenability of G), the two algebras coincide, and our case is one of them. Notice that the cases where $C^*(G)$ is simple are particular examples. These facts are mentioned by informations purposes only and we are not going to develop this program further.

Although the definition of the norm is rather abstract, we have some results to help us compute them

Proposition 3.5. *If $f \in \mathcal{C}_c(G^0)$, then $||f|| = ||f||_\infty$*

We already know that $||f|| \leq ||f||_\infty$ by proposition 3.3. We can present an explicit representation such that $||\pi(f)|| = ||f||_\infty$, and π is exactly the representation π^x already introduced [Lim19].

One more time, we remark that the construction can be generalized for groupoids with less topological properties. Renault, for example, uses an auxiliary norm named $I-$norm to define the definitive norm and it demands some work to show they coincide (one must talk about inductive limit topology, for example).

Example 13. *Let (Σ_n, σ) be the usual shift in n letters. As it is well-known, it is a compact metric space and σ is a local homeomorphism. It is well known that $C^*(G(\Sigma, \sigma)) \approx \mathcal{O}_n$, the Cuntz-algebra. This algebra may be defined by the relations: $\{S_j^* S_i = 1, \sum_i S_i S_i^* = 1; i, j = 1, ..., n\}$.*

3.1.1 Positivity

Most ways to define positivity in a C*-algebra make use of the unit. Unfortunately, a C*-algebra of a groupoid needs not to be unital, although it is a well-known fact that every C*-algebra have an approximate identity (basically, a net of elements such that $a_i x \to x$ for every x). One way to define positivity with no reference to unit is the following.

Definition 3.1. An element f of a C*-algebra $\mathfrak{A}$ is called positive if $f = a^* a$ for some $a \in \mathfrak{A}$. We will say that $f \geq 0$ and denote the set of such elements by $\mathfrak{A}^+$

It is routine to verify that $\mathfrak{A}^+$ is a pointed convex cone, so it defines a order. We also remark that positive elements are self-adjoint, so $\mathfrak{A}$ becomes a *-ordered vector space. We will denote the self-adjoint elements by $\mathrm{Re}(\mathfrak{A})$.

Returning to our case of interest (groupoid algebras), we may recall that $a^* a \in \mathcal{C}_c G^0$, for every $a \in \mathcal{C}_c(\mathcal{U})$, so we may be tempted to think that $C^*(G)^+ \subset \mathcal{C}_c(G^0)$, but it is not true. Indeed, if $f \geq 0$, using partitions of unity:

$$f = a^*a = \left(\sum_j a_j\right)^* \cdot \left(\sum_i a_i\right) = \left(\sum_j a_j^*\right) \cdot \left(\sum_i a_i\right) = \sum_j \sum_i a_j^* \cdot a_i$$

And we have no guarentee the terms $a_j^* \cdot a_i$ for $i \neq j$ are in $\mathcal{C}_c(G^0)$. Concretely, the matrix $f = (22|22)$ is positive (see example 12) since $f = a^*a$ for $a = (11|11)$ and $f \notin \mathcal{C}_c(G^0)$.

Although the cone is not in $\mathcal{C}_c(G^0)$, its restriction to this set is more well-behaved.

Proposition 3.6. *Let $f \in \mathcal{C}_c(G)$. If f is self-adjoint, $f(x) \in \mathbb{R}$, for every $x \in G^0$. If $f > 0$, $f(x) \geq 0$, for every $x \in G^0$. If $f \in \mathcal{C}_c(G^0)$, the reciprocal of the two statements hold.*

Proof. If f is selfadjoint, $f^*(x) = \overline{f(x^{-1})} = f(x)$, for every x. If $x \in G^0$, $x^{-1} = x$, so $\overline{f(x)} = f(x)$. If $f = a^* \cdot a$, then:

$$f(x) = (a^* \cdot a)(x) = \sum_{h \in G_x} \overline{a((xh^{-1})^{-1})} a(h) = \sum_{h \in G_x} \overline{a(h)} a(h) = \sum_{h \in G_x} |a(x)|^2 \geq 0$$

provided $x \in G^0$, because $xh^{-1} = h^{-1}$ in this case.

The reciprocal is evident. If $f > 0$, we may take a to be $a(x) = \sqrt{f(x)}$, and then $f = a^* \cdot a$ by using 3.2. □

3.2 Dynamics

Our purpose is to codify a dynamical system (X, T) in a C*-algebra. We already constructed the framework $C^*(G(X, T))$ and now we are going to spend some time describing ways to transport dynamical quantities from X to $C^*(G)$. In first place, we remark that we may have some function $\varphi \in \mathcal{C}(X, \mathbb{R})$, mainly in mathematical physics, which plays the role of the interaction, or the Hamiltonian. We will call it potential. The object in G which translates the potential will be a 1-cocyle. This section is mainly inspired by [Lim19].

Definition 3.2. Let (X, T) be a dynamical system and $\varphi \in \mathcal{C}(X, \mathbb{R})$. The 1-cocycle in $G(X, T)$ associated to the potential is defined by:

$$c_\varphi(x, m - n, y) = \sum_{i=0}^{m-1} \varphi(T^i(x)) - \sum_{j=0}^{n-1} \varphi(T^j(y))$$

It is matter of worry to verify if c_φ depends only upon the difference $m - n$. It is indeed the case, as a straightforward calculation shows [Lim19]. c_φ is indeed a (continuous) 1-cocycle. Furthermore:

Proposition 3.7. *For every continuous 1-cocycle c on G, there exists a unique continuous function φ such that $c = c_\varphi$*

Proof. See [Lim19] □

Now, we must import this to $\mathcal{C}^*(G)$ in order to make it a C*-dynamical system:

Definition 3.3. Let $\tau = \{\tau_t\}_{t\in\mathbb{R}}$ be an one-parameter group of automorphisms of $\mathfrak{A}$. The group is said strongly continuous if, given $a \in \mathfrak{A}$, the map:

$$t \mapsto \tau_t(a)$$

is norm-continuous. The pair $(\mathfrak{A}, \tau)$ is called a C*-dynamical system.

Proposition 3.8. *A one-paramenter group of *-automorphisms is strongly continuous if, and only if* $||\tau_t(a) - a|| \to 0$, *for every* $a \in \mathfrak{A}$

Example 14. *A strongly continuous one parameter group of *-automorphisms is approximately inner if there is a sequence* $H_n \in Re(\mathfrak{A})$ *such that*

$$\lim_n ||e^{itH_n}Ae^{-itH_n} - \tau_t(A)|| = 0$$

And the limit must be uniform for t in a compact set [PS75]. As usual, this definition may vary among authors. There is another definition of approximately inner which is slightly strong: it requires the limit to converge uniformly in compact sets of $\mathbb{C}$ *for every* A *analytic.*

In our case, as usual, the group of automorphism will be first defined on $\mathcal{C}_c(G)$ and then we will try to extend it.

Definition 3.4. Let c be a 1-cocycle of G. The group of *-automorphisms in $\mathcal{C}_c(G)$ associated to c, denoted by α^c is defined by:

$$(\alpha_t^c(f))(g) = e^{itc(g)}f(g), \quad f \in \mathcal{C}_c(G)$$

Proposition 3.9. α^c *is indeed a group of *-automorphisms*

Proof. It clealy is a linear map, but it is also a *-algebra endomorphism:

$$\alpha_t(f_1 \cdot f_2)(g) = e^{itc(g)}(f_1 \cdot f_2)(g) = e^{itc(g)} \sum_{g_1g_2=g} f_1(g_1)f_2(g_2) = \sum_{g_1g_2=g} e^{itc(g_1g_2)}f_1(g_1)f_2(g_2)$$

$$= \sum_{g_1g_2=g} (e^{itc(g_1)}f_1(g_1))(e^{itc(g_2)}f_2(g_2)) = \sum_{g_1g_2=g} \alpha_t(f_1)(g_1)\alpha_t(f_2)(g_2) = (\alpha_t(f_1) \cdot \alpha_t(f_2))(g)$$

$$\alpha_t(f^*)(g) = e^{itc(g)}f^*(g) = \overline{e^{-itc(g)}f(g^{-1})} = \overline{e^{itc(g^{-1})}f(g^{-1})} = \overline{\alpha_t(f)(g^{-1})} = \alpha_t(f)^*(g)$$

And it is a group:

$$\alpha_t \circ \alpha_s(f)(g) = e^{itc(g)}\alpha_s(f)(g) = e^{itc(g)}e^{isc(g)}f(g) = e^{i(t+s)c(g)}f(g) = \alpha_{t+s}(f)(g)$$

In particular, every endomorphism has a inverse, so they are automorphisms. ☐

Proposition 3.10. *Let $\{\tau\}_t$ be a strongly continuous one-parameter group of *-automorphisms in a dense *-subalgebra A_0 of a C^*-algebra A. If $||\tau_t(a)|| \leq ||a||$, $\forall t \in \mathbb{R}$ and $a \in A_0$, then τ can be uniquely extended to a dynamics in A, which we will call τ as well*

It is obvious that α^c satisfies these requirements, so we finally have a C*-dynamical system associated to (X, T).

Now, some remarks. Recall that every *-automorphism of a C*-algebra is an isometry and that the weak topology is coarser than the norm topology, so every norm-continuous function $\mathbb{R} \to C^*$ is weakly continuous. From this, we conclude that every dynamics τ is a one-parameter weakly continuous group of isometries

Recall that, if X is a topological space and V a normed space, then $f : X \to V$ is weakly continuous if, and only if $\phi \circ f$ is continuous for every continuous function ϕ

Definition 3.5. A subset $X \subset A$ is said $\tau-$invariant if $\tau_t(a) \in X$ for every $a \in X$ and $t \in \mathbb{R}$. In particular, $a \in A$ is $\tau-$invariant if $\{a\}$ is so. A state φ is $\tau-$invariant if $\varphi(\tau_t(a)) = \varphi(a)$ for every $a \in A$ and $t \in \mathbb{R}$

Example 15. *If c is a 1-cocycle in G, then $\mathcal{C}_c(G)$ is α^c-invariant.*

3.3 Some Examples and Useful Constructions

3.3.1 UHF Algebras

UHF is a short for "uniformly hiperfinite". A UHF algebra may be seen as the limit of an increasing sequence of matrix algebras. This type of C*-algebra has a very known theory, and provides interesting examples. The theory was initiated by James Glimm in 1960. An important example of a UHF algebra is the CAR algebra, which is completely related to quantum statistical mechanics: the CAR algebra is a model for a quantum lattice system of spins.

3.3.2 Cuntz-Krieger Algebras and some of their Representations

We will briefly recall some concepts related to isometries in a Hilbert space. An isometric operator is a operator T that preserves distances: $||T(x) - T(y)|| = ||x - y||$. By linearity, $||T(x) - T(y)|| = ||T(x - y)||$. In particular, $||T(x)|| = ||x||$, for every x, and the reciprocal is true again by linearity. That is, if $||T(x)|| = ||x||$ for every vector, T is isometric. This equality means $< T(x), T(x) >=< x, x >$. For being isometric, in particular T is bounded, so it makes sense to talk about adjoint. Then, if T is isometric, $\langle T(x), T(x)\rangle = \langle T^*T(x), x\rangle = \langle x, x\rangle$, for every x. This implies that $T^*T = 1$. The reciprocal is true and easy. Notice it is slightly different from an operator to be unitary. An unitary operator is an operator which is isometric and also bijective. In this case, U has an inverse, so it holds something a little bit stronger: $U^*U = UU^* = 1$.

Now, let G be a closed subspace of a Hilbert space. The orthogonal decomposition theorem states that $H = G \oplus G^{\perp}$, so we can define $P_G(x) = x_G$ such that $x = x_G + x'_G$ such that $x_G \in G$ and $x'_G \in G^{\perp}$. The direct sum assures that this function, called projection is well-defined. It is always true that this function is a bounded, selfadjoint and idempotent operator. Reciprocally, if P is a bounded, sefadjoint and idempotent operator of a Hilbert space, then there exists a closed subspace G such that $P = P_G$, and $G = \mathrm{Im}P(H)$. With this in mind, we say that an element of an arbitrary C^*-algebra is a projection if it is selfadjoint and idempotent. Based on the previous discussion, we also define an isometry to be an element such that $a^*a = 1$ and unitary if $a^*a = aa^* = 1$.

We can relate the two notions in the following statement: if a is a isometry, then aa^* is a projection. aa^* is clearly selfadjoint and $(aa^*)(aa^*) = a(a^*a)a^* = aa^*$.

Now we have the tools to define the Cuntz Algebras. The Cuntz Algebras on n symbols $\{S_1, ..., S_n\}$, denoted $\mathcal{O}_n$ is defined as the universal algebra satisfying the relations: $S_i^* S_i = 1$ (i.e., S_i is an isometry) and $\sum_i S_i S_i^* = 1$. We are going to provide a way to construct these algebras and show explictly this construction (representation) for $\mathcal{O}_2$, in order to turn things clearer.

Let H be an infinite dimensional separable Hilbert space. Then, we can find N infinite dimensional subspaces, orthogonal to each other, such that $H = H_1 \oplus ... \oplus H_n$. Since every infinite-dimensional separable Hilbert space is isometric, we can find n isometric isomorphisms $s_i : H \to H_i$ and we define $S_i : H \to H$ to be s_i but with counterdomain being H. Notice that S_i will also be an isometry (in particular, injective) but not surjective, so it has only left inverses and $S_i^* S_i = 1$, as wanted (S^* is one left inverse). By what was said before, $S_i S_i^*$ are projections to H_i. It is clear that this is a projection and that the image is contained in H_i, but it may also be shown that $S_i S_i^* = 1$ if restricted to H_i, so one can show that $\sum_i S_i S_i^* = 1$ without much work. Notice that $S_i^* x = 0$, for every $x \neq H_i$. In fact, $\langle S_i^* x, y\rangle = \langle x, S_i y\rangle = 0$ for every $y \in H$, because $S_i y \in H_i$, and x is orthogonal to H_i, by construction.

For $n = 2$, let $H = \ell^2(\mathbb{N})$, $H_1 = H_I$ the set of $x \in H$ such that $x_i = 0$ for every i even and $H_2 = H_P$ the set of $x \in H$ such that $x_i = 0$ for every i odd. It is straightforward to see that these sets are subspaces orthogonal to each other and such that $H = H_I \oplus H_P$. It is indeed the case that they are closed. Let $x \notin H_I$, that is, x such that $x_i \neq 0$ for some i even. Take $\epsilon = \sqrt{|x_i|}$. Then, if $y \in B(x, \epsilon)$:

$$d(x, y) < \epsilon \implies \sum_{k \geq 0} |x_k - y_k| < \epsilon^2 \implies |x_i - y_i| < \epsilon^2 \implies |x_i - y_i| < |x_i|$$

$$\implies |y_i| \geq |x_i| - |x_i - y_i| > 0$$

And hence $y \notin H_I$ either. The reasoning is the same for H_P.

Now, let $(e_n)_n$ be the canonical Hilbert basis of $\ell^2(\mathbb{N})$. We define S_1 and S_2 to be such that $S_1(e_n) = e_{2n-1}$ and $S_2(e_n) = e_{2n}$. Furthermore, $S_1^*(e_n) = e_{(n+1)/2}$ if n is odd and 0 otherwise, and $S_2^*(e_n) = e_{n/2}$ if n is even and 0 otherwise. It is straightforward to see that this functions satisfy the defining relations of the Cuntz algebra.

More generally, for $\mathcal{O}_N$, it is possible to define $S_i(e_n) = e_{N(n-1)+i}$.

The defining relations of the Cuntz algebra imply that $S_i^* S_j = 0$ if $i \neq j$. This will be proved in the more general setting of a Cuntz-Krieger algebra that will be introduced next.

3.3.3 Crossed Products

Chapter 4

Aiming the Equilibrium

In this section we are going to investigate some objects related to equilibrium. KMS states are widely known as equilibrium states for quantum statistical mechanics, since the celebrated work by Haag, Hugenholtz and Winnick in 1967. KMS measures are a less known concept, but we establish deep connections between them and the KMS states, justifying the name. Finally, we also consider eigenmeasures (more specifically, fixed points) for the Ruelle's operator. These measures are related to equilibrium states of classical lattice systems since their introduction by Ruelle in 1969. We will also enunciate a relation between them and the KMS measures.

At the end of the day, the KMS measures will play the role of the bridge between KMS states and eigenmeasures (quantum and classical), so we are going to be able to use Ruelle's operator tools to prove results about the existence and uniquness of KMS states.

4.1 KMS states

Given a dynamical system $(\mathfrak{A}, \tau)$, a KMS state ω for τ is a state on A satisfying one of several equivalent conditions and, as already stated, relevant for being related to equilibrium states for quantum statistical mechanics. The conditions generally makes reference to some extension of the dynamics to complex parameters, so we will briefly see some results about it. The first observation is that we wish the extension to be analytic, but we don't have a precise meaning of analytic function with codamin in a C*-algebra. The solution is the content of the next definition. I_λ will denote the set of complex numbers whose imaginary part is less than λ in absolute value.

Definition 4.1. Let $(\mathfrak{A}, \tau)$ be a C*-dynamical system. An element $a \in \mathfrak{A}$ is analytic for τ if there is some $\lambda > 0$ and $f : I_\lambda \to \mathfrak{A}$ such that

$$f(t) = \tau_t(a), \quad t \in \mathbb{R} \text{ and } \eta \circ f \text{ is analytic } , \eta \in \mathfrak{A}'$$

We write

$$\tau_z(a) = f(z)$$

When $I_\lambda = \mathbb{C}$, we say that a is entire. The set of entire elements will be denoted by $\mathfrak{A}_\tau$

This condition is not so restrictive as one could think. As matter of curiosity, for every element of $\mathfrak{A}$, one can find a sequence of entire elements weakly convergent to it. In particular $\mathfrak{A}_\tau$ is weakly dense in $\mathfrak{A}$. Actually, $\mathfrak{A}_\tau$ is even norm dense. It is easy to see that $\mathfrak{A}_\tau$ is a subspace, but yet we have the following:

Proposition 4.1. $\mathfrak{A}_\tau$ *is a norm-dense* $\tau-$*invariant *-subalgebra.*

Now we are ready:

Definition 4.2. Let $(\mathfrak{A}, \tau)$ be a C*-dynamical system and $\beta \in \mathbb{R}$. A state ω is KMS for τ and β if

$$\omega(b\tau_{i\beta}(a)) = \omega(ab)$$

For every $a, b \in A_0$, where A_0 is a $\tau-$invariant norm-dense *-subalgebra of entire elements.

A font of variation in the definition is the set of a, b considered. One may require $b \in \mathfrak{A}$ and $a \in \mathfrak{A}_\tau$ as [KR06], $a, b \in \mathfrak{A}\tau$, a, b in a dense set of analytic elements, and so on. All these are equivalents, and it is even equivalent to consider a, b in an arbitrary dense *-subalgebra. Another useful definition (to prove that KMS states are invariant, for example), is the following:

Proposition 4.2. *ω is a $(\tau, \beta)-$KMS state if, and only if, for any a, b there exists a complex function $F_{a,b}$ analytic in the strip $\{z \in \mathbb{C}; 0 < Im(z) < \beta\}$ and continuous in its closure satisfying:*

$$F_{a,b}(t) = \omega(a\tau_t(b)) \quad F_{a,b}(t + i\beta) = \omega(\tau_t(b)a) \quad \forall t \in \mathbb{R}$$

Proposition 4.3. *Every KMS state is invariant*

In the case of the Reanault-Deaconu groupoid, we have the following result about $C^*(G)_{\alpha^c}$:

Proposition 4.4. $\mathcal{C}_c(G) \subset C^*(G)_{\alpha^c}$

4.2 Quasi-invariant and KMS measures

If we have a measure μ in G^0, it induces two measures in G: μ_r and μ_s, defined by:

$$\mu_r(B) = \int_{G^0} \lambda^x(B) d\mu(x) \quad \mu_s(B) = \int_{G^0} \lambda_x(B) d\mu(x)$$

Formally, λ^x and λ_x are Haar systems, but for our purposes, λ^x may be regarded as the counting measure in G^x and analogously for λ_x on G_x. One can interpret these measures as μ with multiplicities, as the next formula makes explicit:

$$\int f(g)d\mu_r(g) = \int_{G^0} \sum_{g \in G^x} f(g)d\mu(x)$$

Again, there is an analogous formula for μ_s with the natural adaptations. We are mainly concerned in measures such that:

Definition 4.3. A regular measure μ in G^0 is called quasi-invariant when $\mu_r \sim \mu_s$

And finally:

Definition 4.4. A quasi-invariant probability measure μ satisfies the $(\beta, c)-$KMS condition for $\beta \in \mathbb{R}$ and c 1-cocycle when:

$$\frac{d\mu_r}{d\mu_s} = e^{-\beta c}$$

Clearly, we should verify that G is $\sigma-$finite in order to use Radon-Nikodym theorem. In our case, as G is locally compact and second countable, it is $\sigma-$compact. Thus, we have only to show that μ_r and μ_s are finite in compact sets, knowing that μ is. We have:

$$\mu_r(K) = \int_{G^0} f(x)d\mu(x)$$

Where $f(x) = \lambda^x(K)$. It is suffices to show that $f(x)$ is continuous, so it will have a maximum (once it is defined in a compact space) and $\mu_r(K) \leq M\mu(G^0) < +\infty$.

For those not so familiar with the concept of Radon-Nikodym derivative, it says that:

$$\mu_r(B) = \int_B e^{-\beta c(g)} d\mu_s$$

For every measurable set. A well know property of the Radon-Nikodym derivative states that the condition is equivalent to the following equality, for every integrable function:

$$\int f(g)d\mu_r(g) = \int f(g)e^{-\beta c(g)} d\mu_s(g)$$

Remembering the form we can write integral in μ_r and μ_S, we arrive to the fact that μ is a KMS mesure if, and only if:

$$\int_{G^0} \sum_{g \in G^x} f(g)d\mu(x) = \int_{G^0} \sum_{g \in G_x} f(g)e^{-\beta c(g)} d\mu(x)$$

4.3 KMS states and KMS measures

Now we are going to explore the relation between KMS states and KMS measure. In this section we will violently use the Riesz-Markov-Kakutani theorem:

Theorem 4.5 (Riesz-Markov-Kakutani)**.** *Let X be a locally compact Hausdorff topological space. There is a bijection between the regular complex measures in X and $\mathcal{B}(\mathcal{C}_c(X), \mathbb{C})$ given by $\omega_\mu(f) = \int f d\mu$ and a bijection between the regular (real) measures in X and $\mathcal{B}(\mathcal{C}_c(X), \mathbb{R})$*

Recall that our space is locally compact Hausdorff, so, if we have some probability measure μ in G^0 satisfying the KMS condition, we will be able to associate a state ω_μ and that state will be KMS. The first thing we must verify, however, is that ω_μ is indeed a state, not only an arbitrary functional.

Proposition 4.6. *Let $C^*(G)$ be the C*-algebra associated to the groupoid $G(X, T)$, with X being a compact metric space. If μ is a probability measure in G^0, then $\omega = \omega_\mu$ uniquely determines a state of $C^*(G)$*

Proof. In first place, by Riesz-Markov, we may associate $\overline{\omega} \in \mathcal{B}(\mathcal{C}(G^0), \mathbb{R})$ to μ (we are suppong that G^0 is compact, so $\mathcal{C}(G^0)$ is a C*-algebra). We are going to prove first that $\overline{\omega}$ is a state in $\mathcal{C}(G^0)$. Notice that this C*-algebra is unital, with unit being χ_{G^0} so we must only prove that $\overline{\omega}$ is positive and $\overline{\omega}(\chi_{G^0}) = 1$, which is in fact trivial, since μ is a probability and, being $f \in \mathcal{C}(G^0)$ positive, by proposition 3.6, $f(x) \geq 0$, for every $x \in G^0$ and thus, the same Riesz-Markov assures us that $\overline{\omega}(f) \geq 0$, finishing the first part.

Now, let ω be the functional in $\mathcal{B}(\mathcal{C}_c(G), \mathbb{C})$ associated to μ. We are going to write ω in terms of $\overline{\omega}$. As G^0 is clopen, both χ_{G^0} and $\chi_{(G^0)^c}$ are continuous. If $f \in \mathcal{C}_c(G)$, then $f\chi_{G^0}, f\chi_{(G^0)^c} \in \mathcal{C}_c(G)$. Since $f = f\chi_{G^0} + f\chi_{(G^0)^c}$, we have:

$$\omega(f) = \omega(f\chi_{G^0}) + \omega(f\chi_{(G^0)^c}) = \int_{G^0} f|_{G^0} d\mu + \int_{(G^0)^c} f|_{(G^0)^c} d\mu = \int_{G^0} f|_{G^0} d\mu = \overline{\omega}(f|_{G^0})$$

Because the second integral were being evaluated in a null set, $\mu((G^0)^c) = 0$. Thus, if $f \in \mathcal{C}_c(G^0)$ is positive, so is $\omega(f)$, and ω is a positive functional.

By proposition 3.5, one has that $||\chi_{G^0}|| = 1$, and clearly $\omega(\chi_{G^0}) = 1$ for what was discussed, so $||\omega|| \geq 1$. It remains to prove that $||\omega|| \leq 1$.

Finally, as ω is defined in a dense subspace, it can be uniquely extended to a state $\rho \in C^*(G)$ that has the same norm. Besides, ρ is positive: define $f(a) := \rho(a^*a)$ for every $a \in C^*(G)$ and notice that it is continuous, since it is the composition of continuous functions. By the continuity and the fact that $f(a) \geq 0$ in a dense set, we have that $f(a) \geq 0$ always. In other words, $\rho(a^*a) \geq 0$, for every $a \in C^*(G)$, so ρ is positive, and then a state.

□

Proposition 4.7. *Let μ be a probability measure in G^0 and $\omega = \omega_\mu$ the associated state in $C^*(G)$. Then, ω_μ is KMS if, and only if μ is KMS.*

Proof. Suppose that μ is KMS. We are going to verify the KMS condition for $a, b \in \mathcal{C}_c(G)$ first. Using just definitions and basic facts:

$$\omega(b\alpha_{i\beta}(a)) = \int_{G^0} (b\alpha_{i\beta}(a))(x)d\mu(x) = \int_{G^0} \sum_{g_1g_2=x} b(g_1)(\alpha_{i\beta}(a))(g_2)d\mu(x) = \int_{G^0} \sum_{g_1g_2=x} b(g_1)e^{-\beta c(g_2)}a(g_2)d\mu(x)$$

$$\int_{G^0} \sum_{h\in G^{r(x)}} b(h)e^{-\beta c(h^{-1}x)}a(h^{-1}x)d\mu(x) = \int_{G^0} \sum_{h\in G^x} b(h)e^{-\beta c(h^{-1})}a(h^{-1})d\mu(x)$$

And finally using that μ is KMS:

$$\int_{G^0} \sum_{h\in G^x} b(h)e^{-\beta c(h^{-1})}a(h^{-1})d\mu(x) = \int_{G^0} \sum_{h\in G_x} b(h)e^{-\beta c(h^{-1})}e^{-\beta c(h)}a(h^{-1})d\mu(x) = \int_{G^0} \sum_{h\in G_x} b(h)a(h^{-1})d\mu(x) =$$

$$\int_{G^0} \sum_{h\in G_{s(x)}} a(xh^{-1})b(h)d\mu(x) = \int_{G^0} (a\cdot b)(x)d\mu(x) = \omega(ab)$$

And ω is KMS since $\mathcal{C}_c(G)$ is a dense invariant *-subalgebra.

Reciprocally, notice that:

$$\omega(ab) = \int_{G^0} \sum_{h\in G^x} a(h)b(h^{-1})d\mu(x) = \int_{G^0} a(h)b(h^{-1})d\mu_r(h)$$

$$\omega(b\alpha_{i\beta}(a)) = \int_{G^0} \sum_{h\in G_x} a(h)b(h^{-1})e^{-\beta c}d\mu(x) = \int_{G^0} a(h)b(h^{-1})e^{-\beta c}d\mu_s(h)$$

So, for $b = \chi_{G^0}$ the KMS conditin says that:

$$\int_{G^0} a(h)d\mu_r(h) = \int_{G^0} a(h)e^{-\beta c}d\mu_s(h), \quad \forall a \in \mathcal{C}_c(G)$$

And it is enough to state that $\frac{d\mu_r}{d\mu_s} = e^{-\beta c}$ □

4.4 KMS measures and Ruelle eigenmeasures

Lemma 4.8. *μ is KMS if, and only if, μ is a fixed point of the operator $\mathcal{L}^*_{-\beta\phi}$*

Proof. Consider the set $S = \{(y, 1, Ty), y \in X\}$. Then, if $\gamma \in S$, $r(\gamma) = y$ and $s(\gamma) = T(y)$. Thus, ask $T(y) = x$ is the same as asking $s(\gamma) = x$ and $f(y)$ is the same as $(f \circ r)(\gamma)$, so we get the first equality between the integrals below. The second equality comes from the fact that μ is KMS and that $\frac{d\mu_r}{d\mu_s} = (\frac{d\mu_s}{d\mu_r})^{-1}$, so we must add $e^{\beta c_\varphi}$ instead of $e^{-\beta c\varphi}$.

$$\int_X \sum_{Ty=x} f(y)e^{-\beta\phi(y)} d\mu(x) = \int_X \sum_{s(\gamma)=x} (fe^{-\beta\phi} \circ r)(\gamma)1_S(\gamma) d\mu(x) = \int_X \sum_{r(\gamma)=x} (fe^{-\beta\phi} \circ r)(\gamma)1_S(\gamma)e^{\beta c_\phi(\gamma)} d\mu(x)$$

For $\gamma \in S$, $m - n = 1$, so the expression for c_φ turns out to be simple, and one can easily see that $c_\phi(\gamma) = \phi(x) = \phi(r(\gamma))$ what is responsible for cancelling the exponentials and result the first integral below. Finally, it is not hard to show that S have bijection with X so the summation have only one term and we get the final equality.

$$\int_X \sum_{r(\gamma)=x} (f(r(\gamma))1_S(\gamma) d\mu(x) = \int_X f d\mu(x)$$

The converse is very similar. The assumption that μ is a fixed point, with aid of the manipulations made above, gives us the following equality for every $f \in \mathcal{C}_c(G)$.

$$\int_X \sum_{s(\gamma)=x} (fe^{-\beta\phi} \circ r)(\gamma)1_S(\gamma) d\mu(x) = \int_X \sum_{r(\gamma)=x} (fe^{-\beta\phi} \circ r)(\gamma)1_S(\gamma)e^{\beta c_\phi(\gamma)} d\mu(x)$$

$$\Longrightarrow \int_X (fe^{-\beta\phi} \circ r)(\gamma)1_S(\gamma) d\mu_s(\gamma) = \int_X (fe^{-\beta\phi} \circ r)(\gamma)1_S(\gamma)e^{\beta c_\phi(\gamma)} d\mu_r(\gamma)$$

and this implies the Radon-Nikodym derivative to be $e^{\beta c_\varphi}$, as we wanted. □

4.5 Existence and Uniqueness

In this section we enunciate some results due to [KR06] about the existence and uniqueness of KMS states associated to a type of dynamical system. It will be very useful to enunciate an extension of 4.7.

Lemma 4.9. *If $c^{-1}(0)$ is principal, then every KMS ω is of the form ω_μ, with μ probability measure in G^0 satisfying KMS.*

Theorem 4.10. *Let (X, T) be a dynamical system with X compact metric space, and T an exact, expansive local homeomorfism exato and $\varphi \in \mathcal{C}(X, \mathbb{R})$. If α is the group of automorphisms associated to c_φ, then:*

i) There exists a KMS state for β if, and only if $P(T, -\beta\varphi) = 0$

ii) If $P(T, -\beta\varphi) = 0$, $c_\varphi^{-1}(0)$ is principal and φ satisfies Bowen condition, then the state is unique.

Proof. (i) Suppose that ω is a KMS state of $C^*(G)$ and let μ be the measure in G^0 associated by Riesz-Markov with the restriction of ω to $\mathcal{C}(G^0)$. By proposition 4.7, μ is KMS. By lemma 4.8, μ is a fixed point for $\mathcal{L}^*_{\beta\varphi}$. This tells us that $\lambda = 1$ and, so, $P(T, -\beta\varphi) = \log 1 = 0$. The reciprocal is the same path in the other direction: if $P(T, -\beta\varphi) = 0$, there is μ in X fixed point for $\mathcal{L}^*_{\beta\varphi}$, so μ is KMS by lemma 4.8, and the associated state is KMS by 4.7.

(ii) By lemma 4.9, as $c_\varphi^{-1}(0)$, for every KMS state ω we can associate a KMS measure, which is a fixed point for the transfer operator. As φ satisfies Bowen condition, such a measure must be unique, then so is ω. $\square$

Appendix A

Category Theory

A.1 Limits and Colimits

This subject appears here with the excuse of defining a topology on the union of an increasing sequence of sets. To be honest, the inductive limit topology in this special case is very straightforward and needs no appendix. Indeed, the topology in the union will be simply the final topology of the sequence of inclusions.

However, I think the little time extra spent is worthwhile, once the abstract construction is useful in several other situations and can provides us more intuition.

We will start by presenting the notion in a more familiar background, and the generalization will be straightforward. This background will be precisely the category of the sets. Let us take some subcategory of the category of sets, that is, a collection of sets $(X_i)_{i\in I}$ for some index set I (theoretically I must not be a set itself, but it may be useful to think of it this way) and a collection of functions between them such that (i) the collection of functions always include the identity function of a set onto itself $f_{ii} : X_i \to X_i$ and (ii) for each pair of functions, their composition is also in the collection.

The *colimit* of this collection will be a set X with maps $(\varphi_i)_{i\in I}$ such that (i) all the diagrams commute, that is $\varphi_j \circ f_{ij} = \varphi_i$ (we will call this a cone) and such that (ii) if Y is a set and $(\psi_i)_i$ a family that make all the diagrams commute (that is, if (Y, ψ) is a cone), then there exists and it is unique the map $h : X \to Y$ such that $\psi_i = h \circ \varphi_i$.

Let us consider the simplest case where there are no functions between the sets. In this case, the colimit will be simply the disjoint union $X = \bigcup_{i\in I}(\{i\} \times X_i)$ and φ the inclusion maps. Although we are not giving a detailed proof that this is the case, we are going to justify this fact a bit. Without loss of generality, suppose the sets are already disjoint so we may write x_i instead of (i, x_i). Let (Y, ψ) be a cone. We may define $h : X \to Y$ by setting $h(x)$ to be $g_i(x)$, where i is the index of the set X_i where x belong. Notice that, if X was *smaller* than the union, then this index would not be well-defined and such h would not exist. Let us describe this situation in a more concrete way. By *smaller* than the union I mean that we have

$\varphi_i(x_i) = \varphi_j(x_j)$ for some $x_i \in X_i$ and $x_j \in X_j$ (i can be equal to j). In this case, we may come up with a smart cone such that $\psi_i(x_i) = y_i \neq y_j = \psi_j(x_j)$. Thus, whatever h we pick, we will have $(h \circ \varphi_i)(x_i) = (h \circ \varphi_j)(x_j)$, so it will be impossible that $\psi_i = h \circ \varphi_i$ and $\psi_j = h \circ \varphi_j$ simultaneously. Ok, but what if X were *bigger* than the union of the sets? Well, just like the smallness of X can spoil the existence of h, the bigness can spoil the uniqueness. If we pick X which has some x coming out of the blue, then we will be able to define $h_1(x)$ and $h_2(x)$ in different ways in such a way that both h_1 and h_2, although different function,s will satisfy the necessary requirements. We will have $\psi_i = h_1 \circ \varphi_i$ and $\psi_i = h_2 \circ \varphi_i$ because x is not in the domain, so it will not matter. In this example we may realize a general property: the colimit X will be unique up to isomorphism.

This simplest case is known as "sum" or "coproduct" in category theory. Giving another example, the coproduct in the category of the vector spaces is the direct sum.

Now, let us consider a little harder case, that will be more useful for us. If, in addition, we require the property that, for every pair of sets X_i, X_j, there exists a set X_k and functions $f_{ik} : X_i \to X_k$ and $f_{jk} : X_j \to X_k$, then the colimit will be called an *inductive limit* or *direct limit*. In this case, we may view I as a directed set and $f_{ij} : X_i \to X_j$ always that $i \leq j$.

In this case, the existence of a function $f : X_i \to X_j$ will impose a severe restriction in the collection (Y, ψ). Suppose that $f(x_i) = x_j$. For being a cone, we will necessarily have $\psi_i(x_i) = \psi_j(f(x_i)) = \psi_j(x_j)$. In other words, $\psi_i(x_i) = \psi_j(x_j)$ for every cone, so there is no point to consider x_i and x_j as different things anymore! We will not be able to come up with a smart cone to separate those points, as we did before, and X will have to be like a "union" but with x_i and x_j being identified. This is the idea behind the general construction of an inductive limit that will be described in what follows.

X will be the disjoint union with the quotient given by the relation: $x_i \sim x_j$ for $x_i \in X_i$ and $x_j \in X_j$ if there is $k \geq i$ and $k \geq j$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. The maps φ_i is the composition of the quotient map with the inclusion.

In the case where the net is non-decreasing, that is $X_i \subset X_j$ whenever $i \leq j$, then the inductive limit is the union $\bigcup_i X_i$ with the maps φ_i being the inclusion.

If we invert the direction of the morphisms, we will have the definition of a limit insteas of a colimit and of a inverse limit instead of a direct limit. In the case where we don't have any function, the limit will be the cartesian product instead of the union.

The colimit will be the initial object in the category of the cocones.

A.2 Free Objects

In mathematics, we often seek to produce algebraic structures that are the largest ones that can be generated by a certain set S, where "largest ones" means that there are no (unnecessary) relations between the elements of the set. These relations would shrink the set of possible elements that can be derived by the elements in S. That is, the only relations the elements satisfy are strictly the relations imposed by the algebraic structure.

For example, if the structure is of vector space, the condition of no unnecessary relations means that the vectors have to be linearly independent. For example, $v + w = w + v$ is a necessary relation between elements, due to the very definition of vector space, but $v + w = 0$ is not. Thus, we are looking for a space that is generated by a certain set that is linearly independent, which is equivalent of saying that the set is a basis for the space. If S is a set, for example $S = \{x_1, x_2, x_3\}$, the free vector space generated by S is the set of all formal linear combinations of S that satisfy only the conditions of vector space and S will be a basis for that space. This thinking generalizes in obvious ways to groups, for example.

Now, let's try to think of a precise way of defining these free objects. Taking advantage of the example of vector spaces, we recall a useful property that characterizes a basis:

Proposition A.1. *Let V be a vector space, $\mathcal{B}$ a basis for V and $f : \mathcal{B} \to W$ any function with W vector space. Then, there exists a linear transformation $T : V \to W$ that extends f, that is, $T|_{\mathcal{B}} = f$. This transformation is unique: if U is linear that coincides with f in $\mathcal{B}$, then $T = U$. Moreover, there is am explicit formula for T. If v is written in terms of the elements of the basis as $\sum a_i v_i$, then:*

$$T(v) = \sum a_i f(v_i)$$

as would have to be by linearity. Notice that this is well defined since there is only one way to write a vector in term of basis elements.

This proposition tells us two things: in first place, that linear transformations are not so rigid, in the sense that we have the freedom to choose the values it will have in any set that is contained in some basis. On the other hand, it is not so free in the sense that a basis is the maximum set that we have this freedom — if we choose the values in a bigger set, there is no guarantee that such linear transformation exists.

From another point of view, a basis is not big enough to not gives us total freedom about the values of linear transformations in its elements and is not small enough to give rise to more than one linear transformation. The first property is due to the fact that elements of a basis is sufficiently independent so values of T in one element don't affect the value in other elements. The second property comes from the fact that every element of V can be generated by elements of the basis.

This characterization is precisely what we want. With this idea in mind, we are going to make a template to define free objects. Let S be a set. A $\langle$structure$\rangle$ F is said a free $\langle$structure$\rangle$ generated by S if there is a function $i : S \to F$ such that, for every $\langle$structure$\rangle$ Y and every function $f : S \to Y$ there is a unique $\langle$morphism$\rangle$ $\phi : F \to Y$ such that the diagram commutes, that is, $\phi \circ i = f$. Notice that $\langle$structure$\rangle$ may be replaced by "group", "ring", "vector space", "algebra" and so on, while $\langle$morphism$\rangle$ is replaced by the respective homomorphism.

Now, how can we write this within the language of categories?

In first place, notice that, besides talking about structures and morphisms, we are dealing with sets and functions, since S has no structure whatsoever, so there is more than one category in play. The relationship between two categories is provided by a functor. Thus, let $F : \text{Set} \to \text{Cat}$ be a candidate for functor, where Set is the category of sets and Cat is any category, and we want $F(S)$ be the free object in Cat generated by S. Let's think what property this functor must satisfy.

In first place, we need a function between S and $F(S)$, that we have denoted by i. This is not directly transposed to categories because S and $F(S)$ lives in different categories. As i is a simple function, we have to bring $F(S)$ back to *Set* and talk about morphisms in this category. The way to do that is the most obvious one: we just take the underlying set of $F(S)$ and forget about the additional structures. For obvious reasons, this functor is called the forgetful functor, and will be denoted by $G : \text{Cat} \to \text{Set}$. The same occur for the function f, that must be between S and $G(V)$.

Thus, we may say that a functor $F : \text{Set} \to \text{Cat}$ is a free functor if, for every $S \in \text{Set}$, for every $W \in \text{Cat}$ and every $f : S \to G(W)$ there exists a morphism i from S to $G(F(V))$ and a morphism $\phi : F(V) \to W$ such that $f = G(\phi) \circ i$. We then say that F is left adjoint to G.

There is yet another way to characterize free objects using categories. Notice that the statement "given $S \in \text{Set}$ and $V \in \text{Cat}$, for every morphism (function) between S and $G(V)$ there is a unique morphism between $F(S)$ and V" means that there is a bijection between $\text{Hom}(S, G(V))$ and $\text{Hom}(F(S), V)$. This may be seen as the definition of F and G being adjoints.

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